Lesson - Trigonometry in right-angled triangles

Year: Year 9

This lesson covers naming the sides, the sine, cosine and tangent ratios, finding missing sides and angles, and trigonometry in context.

Starter activity

Draw a right-angled triangle with an angle of 40° and a hypotenuse of exactly 10 cm. Measure the opposite and adjacent sides. Divide the opposite side by the hypotenuse. Repeat with a hypotenuse of 20 cm. What do you notice about your ratios? Compare with $\sin 40^\circ$ on a calculator.

Why trigonometry?

Note: sides and angles

Pythagoras' theorem links the three *sides* of a right-angled triangle. **Trigonometry** links the *sides and the angles*. In **similar triangles** (the same angles) the ratios between sides are always the same, so each angle has its own set of fixed ratios. Surveyors, pilots and engineers use it every day.

Naming the sides

Definition: hypotenuse, opposite and adjacent

Label the sides in relation to the angle θ you are interested in:

- the **hypotenuse (H)** is opposite the right angle, the longest side;
- the **opposite (O)** is the side across from θ;
- the **adjacent (A)** is the side next to θ (not the hypotenuse).

The opposite and adjacent sides swap if you use the other acute angle, so always label for the angle in the question.

The three ratios

Rule: sine, cosine and tangent

$$\sin\theta = \frac{O}{H} \qquad \cos\theta = \frac{A}{H} \qquad \tan\theta = \frac{O}{A}$$

A memory aid is **SOH CAH TOA**. Make sure your calculator is in **degree mode** (it

shows D or DEG).

Key fact: values worth knowing

$\sin 30^\circ = 0.5$, $\cos 60^\circ = 0.5$ and $\tan 45^\circ = 1$.

Finding a missing side

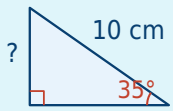
Method: finding a missing side

1. Sketch the triangle and label H, O and A for the angle given.
2. Decide which two sides are involved (one known, one unknown) and pick the ratio that uses them.
3. Write the equation, rearrange and calculate. When the unknown is on the *top* of the fraction you multiply; when it is on the *bottom* you divide.

Worked example 1: using sine

A right-angled triangle has an angle of 35° and a hypotenuse of 10 cm. Find the side opposite the angle.

We know H and want O, so use sin: $\sin 35^\circ = \dfrac{x}{10}$, $x = 10 \sin 35^\circ = 5.74$ cm to 3 s.f.

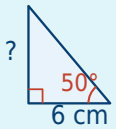


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Worked example 2: using tangent

Angle 50°, adjacent 6 cm. Find the opposite side.

We know A and want O: $\tan 50^\circ = \dfrac{x}{6}$, $x = 6 \tan 50^\circ = 7.15$ cm.

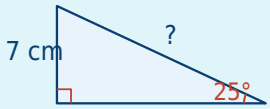


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Worked example 3: unknown on the bottom

Angle 25°, opposite 7 cm. Find the hypotenuse.

$\sin 25^\circ = \frac{7}{x}$. The unknown is on the bottom, so swap: $x = \frac{7}{\sin 25^\circ} = 16.6$ cm.



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Finding a missing angle

Method: finding a missing angle

1. Label the sides and choose the ratio for the two known sides.
2. Write $\sin\theta$, $\cos\theta$ or $\tan\theta$ equal to the fraction.
3. Use the **inverse function** ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, usually with the SHIFT key) to find θ.

Worked example 4: inverse sine

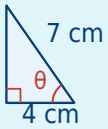
Opposite 3 cm, hypotenuse 5 cm. Find the angle.

$\sin\theta = \frac{3}{5} = 0.6$, so $\theta = \sin^{-1}(0.6) = 36.9^\circ$.

Worked example 5: inverse cosine

Adjacent 4 cm, hypotenuse 7 cm. Find the angle.

$\cos\theta = \frac{4}{7}$, $\theta = \cos^{-1}\left(\frac{4}{7}\right) = 55.2^\circ$.

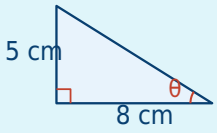


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Worked example 6: inverse tangent

Opposite 5 cm, adjacent 8 cm. Find the angle.

$\tan\theta = \frac{5}{8} = 0.625$, $\theta = \tan^{-1}(0.625) = 32.0^\circ$.



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Problems in context

Definition: angle of elevation

The **angle of elevation** is the angle measured upwards from the horizontal.

Worked example 7: elevation, ladder, isosceles triangle

(a) A tower is 30 m tall and you stand 45 m from its base: find the angle of elevation. (b) A 6 m ladder makes an angle of 70° with the ground: find the height reached and the distance of the foot from the wall. (c) An isosceles triangle has base 10 cm and equal sides 13 cm: find a base angle.

(a) The **angle of elevation** is measured upwards from the horizontal. $\tan\theta = \dfrac{30}{45}$, so $\theta = \tan^{-1}(0.667) = 33.7°$. (b) Height $6\sin 70° = 5.64$ m and foot $6\cos 70° = 2.05$ m from the wall; check with Pythagoras: $5.64^2 + 2.05^2 \approx 36$. (c) Split it down the middle to get a right-angled triangle with adjacent 5 and hypotenuse 13: the base angle is $\cos^{-1}\left(\dfrac{5}{13}\right) = 67.4°$.

Checking your answer

Note: is the answer sensible?

The hypotenuse must be the longest side, so any leg you find must be shorter than it. An acute angle must be between 0° and 90°, and the two acute angles add to 90°, so the other angle is $90° - \theta$. Write down the full calculator display before rounding, and round to 3 significant figures for lengths and 1 decimal place for angles unless told otherwise. If you know two sides and want the third, use Pythagoras; if an *angle* is involved, use trigonometry.

Watch out: common mistakes

- Having the calculator in radians instead of degrees.
- Labelling opposite and adjacent from the wrong angle.
- Multiplying instead of dividing when the unknown is the denominator.
- Using $\sin$ instead of $\sin^{-1}$ when finding an angle.

- Trying to use trigonometry in a triangle without a right angle.

Worked example 8: finding the hypotenuse

A ramp rises 1.2 m at an angle of 15° to the ground. Find the length of the ramp.

The rise is opposite the angle and we need the hypotenuse, so use sine: $\sin 15^\circ = \dfrac{1.2}{L}$. Rearranging, $L = 1.2 \div \sin 15^\circ = 4.64$ m (to 2 decimal places).

Worked example 9: the angle of depression

From the top of a cliff 50 m high, the angle of depression of a boat is 18°. How far is the boat from the base of the cliff?

The angle of depression equals the angle of elevation from the boat to the top, by alternate angles. $\tan 18^\circ = \dfrac{50}{d}$, so $d = 50 \div \tan 18^\circ = 153.9$ m.

Worked example 10: two triangles

In triangle ABC, angle $B = 90^\circ$, $AB = 8$ cm and angle $A = 35^\circ$. Find BC and then AC.

$BC = 8 \tan 35^\circ = 5.60$ cm. $AC = 8 \div \cos 35^\circ = 9.77$ cm. Check with Pythagoras: $\sqrt{8^2 + 5.6^2} = \sqrt{95.36} = 9.77$.

Worked example 11: choosing the right ratio

A kite string is 25 m long and the kite is 18 m above the ground. Find the angle the string makes with the ground.

We know the opposite (18 m) and the hypotenuse (25 m), so use sine.
$\sin \theta = \dfrac{18}{25} = 0.72$, so $\theta = \sin^{-1} 0.72 = 46.1^\circ$.

Note: how to choose the ratio

Label the sides O, A and H for your angle. Look at which two sides are involved (one known and one unknown). If they are O and H use sine, if A and H use cosine and if O and A use tangent. Write SOH CAH TOA at the top of your page to help you remember it.

Watch out: calculator mode and rounding

- Check that the calculator is in degrees (DEG), not radians. $\sin 30^\circ$ must give 0.5.

- When the unknown is on the bottom of the fraction you divide: $x = 7 \div \tan 40^\circ$, not $7 \times \tan 40^\circ$.
- The hypotenuse is always the longest side. If your answer for a shorter side is longer than the hypotenuse, you have made an error.

Did you know?

Trigonometry began with astronomers, who measured angles between stars. The word sine comes from a mistranslation: an Indian word meaning a bowstring was borrowed into Arabic and then mis-read in Latin as the word for a fold or a bay, which is sinus. Today the same ideas help GPS satellites to find your position.

Trigonometry step by step

Worked example 12: a complete solution

A ramp rises 1.2 m over a horizontal distance of 9 m. Find the angle of the ramp to the horizontal, to 1 decimal place.

First sketch the right-angled triangle and mark the angle θ at the bottom of the ramp. The side of 1.2 m is opposite θ and the side of 9 m is adjacent to it. The hypotenuse is not known and not needed, so the ratio to use is tangent. Write $\tan\theta = \dfrac{1.2}{9} = 0.1333\ldots$ Then use the inverse function: $\theta = \tan^{-1}(0.1333\ldots) = 7.594\ldots^\circ$, which is 7.6° to 1 decimal place. A gentle slope should have a small angle, so the answer is sensible.

Method: setting out a trigonometry answer

1. Draw and label a sketch, marking the right angle and the angle you are working with.
2. Label the three sides H, O and A relative to that angle.
3. Tick the two sides that are involved (the one you know and the one you want, or the two you know when finding an angle).
4. Choose SOH, CAH or TOA, write the equation in full, and only then rearrange it.
5. Keep the full calculator value until the end, round as asked, and give the units: centimetres, metres or degrees.

Note: trigonometry or Pythagoras?

If a question involves only the three sides of a right-angled triangle, use Pythagoras' theorem. If it involves an angle as well as sides, either because an angle is given or because you need to find one, use trigonometry. In longer problems you may need both: for example, use Pythagoras to find a missing side, then trigonometry to find an angle.

Otto's tip

Write O, A, H on your sketch before choosing the ratio. Then circle the two letters you need and match them with SOH CAH TOA. Hoot hoot!