Lesson - Similarity and enlargement

Year: Year 9

This lesson covers congruent triangles, scale factors, missing lengths in similar shapes, testing for similarity and enlargement on a coordinate grid.

Starter activity

Look at a photograph on a screen and drag one corner to make it bigger while keeping the proportions locked. Then try dragging only the width. What happens to the people in the picture? Write down what has to stay the same when a picture is enlarged without being distorted.

Same shape, different size

Definition: congruent and similar

Two shapes are **congruent** if they are exactly the same shape and size, although one may be rotated, reflected or moved. Two shapes are **similar** if they have the same shape but different sizes: all their angles are equal and all their corresponding sides are in the same ratio. A photograph and its enlargement, or a model car and the real car, are similar.

Congruent triangles

Rule: congruence conditions

Two triangles are congruent if one of these conditions holds:

- **SSS**: all three sides are equal;
- **SAS**: two sides and the angle *between* them are equal;
- **ASA** (or AAS): two angles and a corresponding side are equal;
- **RHS**: both triangles are right-angled, with equal hypotenuses and one other equal side.

Watch out: AAA

Triangles with sides 5, 6, 7 and 7, 5, 6 are congruent by SSS. But AAA (three equal angles) only proves the triangles are *similar*, not congruent.

Scale factors

Definition: enlargement and scale factor

An **enlargement** changes the size of a shape. The **scale factor** tells you how many times bigger each length becomes:

$$\text{scale factor} = \frac{\text{new length}}{\text{old length}}$$

A scale factor greater than 1 makes the shape bigger; between 0 and 1 (for example $\frac{1}{2}$) it makes it smaller. Angles never change.

Worked example 1: enlarging a rectangle

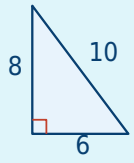
A 3 cm by 5 cm rectangle is enlarged with scale factor 2. Find the new perimeter and area.

The new rectangle is 6 cm by 10 cm. The perimeter doubles from 16 cm to 32 cm. The area goes from 15 cm² to 60 cm², which is multiplied by $2^2 = 4$.

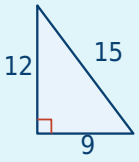
Worked example 2: similar triangles

Two triangles are similar. In the first the sides are 6, 8 and 10. In the second the shortest side is 9. Find the other two sides.

The scale factor is $9 \div 6 = 1.5$. The other sides are $8 \times 1.5 = 12$ and $10 \times 1.5 = 15$.



Not drawn accurately



Not drawn accurately

Missing lengths in similar shapes

Method: missing lengths in similar shapes

1. Match up corresponding sides (the sides that are in the same position).
2. Find the scale factor using a pair of sides where both lengths are known.
3. Apply it to the length you want: *multiply* to go from the small shape to the big one, and *divide* to go from big to small.

Worked example 3: a missing side

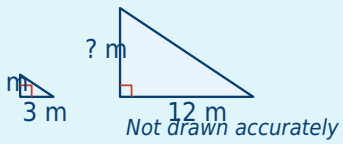
Two similar shapes have corresponding sides of 4 cm and 6 cm. A second side is 10 cm in the smaller shape. Find the corresponding side in the larger shape.

The scale factor is $6 \div 4 = 1.5$. The side is $10 \times 1.5 = 15$ cm.

Worked example 4: shadows and models

A pole 2 m tall casts a shadow of 3 m. At the same moment a tree casts a shadow of 12 m. How tall is the tree? A model at scale 1 : 50 is 12 cm high; what is the real height?

The triangles are similar and the scale factor is $12 \div 3 = 4$, so the tree is $2 \times 4 = 8$ m tall. The model height represents $12 \times 50 = 600$ cm, or 6 m.



Are two shapes similar?

Key fact: testing for similarity

For polygons, check that every pair of corresponding sides has the same ratio. Rectangles 4 × 6 and 6 × 9 are similar because $6 \div 4 = 1.5$ and $9 \div 6 = 1.5$. Rectangles 3 × 5 and 6 × 9 are not: $6 \div 3 = 2$ but $9 \div 5 = 1.8$. For triangles it is enough to show that two pairs of angles are equal, because the third pair must then be equal as well. A triangle with angles 35° and 75° has a third angle of 70°, so it is similar to a triangle with angles 75° and 70°.

Enlargement on a coordinate grid

Rule: describing and drawing an enlargement

To describe an enlargement give the **scale factor** and the **centre of enlargement**. To draw one, multiply the distance of each point from the centre by the scale factor, along the same direction.

Worked example 5: centre at the origin

Enlarge $(2, 1)$ by scale factor 3, and triangle $A(1, 1)$, $B(3, 1)$, $C(1, 4)$ by scale factor 2.

Multiply both coordinates by the scale factor. $(2, 1)$ goes to $(6, 3)$. The triangle goes to $A'(2, 2)$, $B'(6, 2)$, $C'(2, 8)$.

Worked example 6: another centre and a fraction

Enlarge $(3, 2)$ by scale factor 2 with centre $(1, 1)$. Then enlarge $(8, 4)$ by scale factor $\frac{1}{2}$ from the origin.

The vector from the centre to $(3, 2)$ is $(2, 1)$. Doubling gives $(4, 2)$, so the image is $(1 + 4, 1 + 2) = (5, 3)$. With scale factor $\frac{1}{2}$, $(8, 4)$ goes to $(4, 2)$: the shape gets smaller but is still called an enlargement.

Watch out: common mistakes

- Adding the scale factor instead of multiplying (adding 2 is not the same as an enlargement of scale factor 2).
- Matching the wrong corresponding sides.
- Dividing when you should multiply, or the other way round: check that a bigger shape gives a bigger answer.
- Forgetting that the area scales by the square of the scale factor, not by the scale factor itself.
- Giving a description of an enlargement without the centre.

Worked example 7: areas of similar shapes

Two similar posters have widths 20 cm and 50 cm. The smaller has an area of 300 cm^2. Find the area of the larger.

Scale factor $50 \div 20 = 2.5$. The area scale factor is $2.5^2 = 6.25$. Area of the larger poster $= 300 \times 6.25 = 1875$ cm^2. The area is multiplied by the *square* of the length scale factor.

Worked example 8: a negative or fractional scale factor

Enlarge triangle with vertices $(2, 4)$, $(6, 4)$ and $(2, 8)$ by scale factor $\frac{1}{2}$ with the centre at the origin.

Multiply each coordinate by $\frac{1}{2}$: $(1, 2)$, $(3, 2)$ and $(1, 4)$. The image is half as long in every direction and is closer to the origin. A fractional scale factor makes the shape smaller.

Worked example 9: finding the scale factor from corresponding sides

Triangles ABC and PQR are similar. $AB = 8$ cm corresponds to $PQ = 14$ cm. Find QR if $BC = 12$ cm.

Scale factor $14 \div 8 = 1.75$. $QR = 12 \times 1.75 = 21$ cm. Check using another pair: if $AC = 10$ cm then $PR = 17.5$ cm.

Worked example 10: describing an enlargement

Triangle $A(1,1), B(2,1), C(1,3)$ is mapped to $A'(3,3), B'(6,3), C'(3,9)$. Describe the transformation.

Each coordinate has been multiplied by 3, so it is an enlargement with scale factor 3. Since $A(1,1)$ goes to $(3,3)$, the lines joining each point to its image meet at the origin, so the centre is $(0,0)$.

Note: proving that triangles are similar

You do not need to check all the sides. If two triangles have two pairs of equal angles, the third pair is also equal, so they are similar. Or check that the sides are in the same ratio. Write the vertices in matching order, so ABC and PQR means that A matches P, B matches Q and C matches R.

Watch out: area and volume

- For lengths use the scale factor k. For areas use k^2. For volumes use k^3.
- A scale factor between 0 and 1 gives a smaller shape. Do not call this a "reduction": it is still an enlargement.
- Equal angles in the same order only prove similarity, never congruence.

Did you know?

Architects and film-makers use similarity all the time. The film-makers who made early monster movies used tiny scale models of cities, but they had to film them at a high speed so that the falling buildings seemed to fall at the right speed: a model at scale 1 : 25 has to be filmed so that time is stretched by about 5 times.

Similarity in everyday life

Worked example 11: scale drawings

An architect draws a kitchen at a scale of 1 : 50. On the drawing the kitchen is 9 cm long. How long is the real kitchen?

A scale of 1 : 50 means that every length in real life is 50 times the length on the drawing, so the scale factor from the drawing to the kitchen is 50. Multiply: $9 \times 50 = 450$ cm, which is 4.5 m. To go the other way, from real life to the drawing, divide by 50. A worktop 3 m long is 300 cm, so it is drawn as $300 \div 50 = 6$ cm. Always change the real length into the same unit as the drawing before you divide.

Method: matching corresponding sides

1. Redraw the two shapes separately, the same way round, if they overlap or one is turned over.
2. Mark equal angles; the sides opposite equal angles correspond.
3. Find one pair of corresponding sides where both lengths are known and divide to get the scale factor.
4. Multiply to find a side in the bigger shape, or divide to find a side in the smaller shape.
5. Check that your answer is sensible: sides of the bigger shape must be longer.

Otto's tip

Always write “scale factor = new ÷ old” at the start. Then check your answer: if the shape got bigger, the lengths must be larger. Hoot hoot!