Lesson - Sequences and nth term

Year: Year 9

This lesson covers term-to-term and position-to-term rules, arithmetic and geometric sequences, special sequences, quadratic sequences and sequences in patterns.

Starter activity

Lay out a row of squares using matchsticks. One square needs 4 matchsticks, two squares need 7, three squares need 10. Without counting, how many matchsticks do 5 squares need? How many do 20 need? Can you find a rule that tells you the number for any number of squares?

What is a sequence?

Definition: sequence, term and position

A **sequence** is a list of numbers that follows a rule. Each number is called a **term**. The first term is $T(1)$ (or u_1), the second is $T(2)$, and so on. The **position** of a term is called n.

Two kinds of rule

Definition: term-to-term and position-to-term rules

A **term-to-term rule** tells you how to get from one term to the next: for 4, 7, 10, 13, ... the rule is “add 3”. Its weakness is that to find the 100th term you would have to work out the first 99. A **position-to-term rule** (the nth term) gives each term directly from its position.

Worked example 1: using an nth term

Find the first four terms and the 50th term of the sequence $3n + 1$.

The terms are $3 \times 1 + 1 = 4$, $3 \times 2 + 1 = 7$, 10 and 13. The 50th term is $3 \times 50 + 1 = 151$.

Arithmetic sequences

Definition: arithmetic sequence

An **arithmetic sequence** goes up or down by the same amount each time. This amount is the **common difference** d.

Rule: nth term of an arithmetic sequence

$$T(n) = dn + (a - d)$$

where a is the first term and d is the number in front of n.

Method: finding the nth term

1. Find the common difference d. This is the number in front of n.
2. Write down dn, the d times table.
3. Compare dn with the sequence and work out the adjustment (what you add or subtract).

Worked example 2: increasing sequence

Find the nth term of 5, 9, 13, 17, ...

The difference is 4, so start with $4n$: 4, 8, 12, 16. Each term of the sequence is 1 more, so the nth term is $4n + 1$.

Worked example 3: decreasing sequence

Find the nth term of 20, 17, 14, 11, ...

The difference is -3, so start with $-3n$: $-3, -6, -9, -12$. To get from -3 to 20 we add 23, so the nth term is $-3n + 23$, also written $23 - 3n$.

Worked example 4: is it a term?

Is 100 a term in the sequence $3n + 1$? Is 50 a term of $4n + 1$?

$3n + 1 = 100$ gives $3n = 99$, so $n = 33$. Since n is a whole number, 100 **is** the 33rd term. $4n + 1 = 50$ gives $n = 12.25$, not a whole number, so 50 is **not** a term.

Geometric sequences

Definition: geometric sequence

A **geometric sequence** is multiplied by the same number each time, the **common ratio** r. For 2, 6, 18, 54, ... the ratio is 3, and the next term is $54 \times 3 = 162$. For 80, 40, 20, 10, ... the ratio is $\frac{1}{2}$: the sequence decreases but never reaches zero.

Other special sequences

Key fact: special sequences

- **Square numbers:** 1, 4, 9, 16, 25, ... with nth term n^2. The differences, 3, 5, 7, 9, ..., go up by 2 each time.
- **Triangular numbers:** 1, 3, 6, 10, 15, ... with nth term $\frac{n(n+1)}{2}$. The 10th term is $\frac{10 \times 11}{2} = 55$.
- **Fibonacci-type sequences:** each term is the sum of the previous two: 1, 1, 2, 3, 5, 8, 13, 21, ...
- **Cube numbers:** 1, 8, 27, 64, ... with nth term n^3.

Worked example 5: a quadratic sequence

Find the nth term of 2, 5, 10, 17, ...

The first differences are 3, 5, 7 and the second differences are 2, 2. A sequence is **quadratic** if the second differences are constant, and the coefficient of n^2 is half the second difference, here 1. Comparing with n^2: 1, 4, 9, 16 shows each term is 1 more, so the nth term is $n^2 + 1$.

Sequences in patterns

Worked example 6: matchstick patterns

A pattern uses 4 matchsticks for shape 1, 7 for shape 2 and 10 for shape 3. Find the nth term, the matches for shape 20, and which shape uses exactly 100.

Each shape adds 3 matches so the nth term is $3n + 1$. Shape 20 uses $3 \times 20 + 1 = 61$ matchsticks. For 100: $3n + 1 = 100$, $n = 33$, so shape 33 uses exactly 100.

Worked example 7: a rule from two terms

An arithmetic sequence has 5th term 23 and 8th term 35. Find the nth term.

The three steps from term 5 to term 8 add $35 - 23 = 12$, so $d = 4$. The nth term is $4n + c$ with $4 \times 5 + c = 23$, so $c = 3$. The nth term is $4n + 3$.

Watch out: common mistakes

- Writing the nth term as $n + 3$ when the difference is 3 (it should be $3n + \ldots$).
- Getting the adjustment sign wrong: compare dn with the first term.
- Assuming a sequence is arithmetic without checking all the differences.
- Saying a term belongs to the sequence without checking that n is a whole number.

Worked example 8: finding when a term passes a value

The nth term of a sequence is $7n - 4$. Which is the first term that is greater than 100?

Solve $7n - 4 > 100$: $7n > 104$, so $n > 14.86$. The first whole number is $n = 15$. The 15th term is $7 \times 15 - 4 = 101$. Check: the 14th term is 94, which is less than 100.

Worked example 9: a decreasing sequence

Find the nth term of $50, 43, 36, 29, \ldots$

The common difference is -7, so the nth term starts with $-7n$. When $n = 1$, $-7n = -7$, and we need 50, so we add 57. The nth term is $57 - 7n$. Check: $n = 4$ gives $57 - 28 = 29$.

Worked example 10: sums of two sequences

Sequence P has nth term $2n + 1$ and sequence Q has nth term $3n - 1$. Find the nth term of the sequence made by adding corresponding terms.

$(2n + 1) + (3n - 1) = 5n$. So the new sequence is $5, 10, 15, 20, \ldots$ with nth term $5n$.

Worked example 11: a geometric sequence in context

A new phone app has 500 users in week 1, and the number grows by 20% each week. How many users are there in week 5?

The ratio is 1.2. Week 5 is 4 steps after week 1: $500 \times 1.2^4 = 500 \times 2.0736 = 1036.8$, so about 1 037 users.

Note: checking your nth term

Always check a formula for the nth term with at least two of the given terms. Substitute $n = 1$ to see whether you get the first term and $n = 3$ for the third. A quick error check is that the number in front of n must equal the common difference. If you have the wrong number there, your formula cannot be right.

Watch out: positions and values

- n is the *position* of the term and the answer is the *value*. The 10th term uses $n = 10$.
- For a number to be in a sequence, n must be a positive whole number.
- Do not call a sequence arithmetic unless the difference is the same every time, and geometric only if the ratio is the same every time.

Did you know?

The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13 and so on appears in nature: many flowers have a number of petals that is a Fibonacci number, and sunflower seeds spiral in 34 and 55 curves. The sequence is named after Leonardo of Pisa, who described it in a book of 1202.

Recognising and explaining sequences

Method: what type of sequence is it?

1. Work out the differences between consecutive terms. If they are all the same, the sequence is arithmetic and its nth term has the form $dn + c$.
2. If the differences change, divide each term by the one before it. If the ratio is always the same, the sequence is geometric.
3. If neither works, find the second differences. If they are constant, the sequence is quadratic and its nth term contains n^2.
4. Otherwise, compare the terms with the special sequences: square, cube and triangular numbers, or a Fibonacci-type rule.

Worked example 12: explaining a formula from a pattern

Hexagons are joined in a row. One hexagon needs 6 sticks, two need 11 and three need 16. Explain why the nth term is $5n + 1$.

The differences are all 5, so the nth term starts with $5n$, and the first term $6 = 5 + 1$ gives $5n + 1$. The pattern explains why. The first hexagon needs one extra stick on its left side to close it. After that, each new hexagon shares one side with the hexagon before it, so it only needs 5 new sticks. Altogether there is 1 starting stick and 5 sticks for each of the n hexagons. For 30 hexagons you would need $5 \times 30 + 1 = 151$ sticks.

Note: sequences in computing

Computer programs often build sequences using a loop that repeats a term-to-term rule, such as "add 4" or "multiply by 2". A position-to-term rule is faster, because it can jump straight to the millionth term without working out every term before it.

Otto's tip

Test your nth term by substituting $n = 1$ and $n = 2$. If you get the first two terms, you are almost certainly right. Hoot hoot!