Lesson - Pythagoras' theorem

Year: Year 9

This lesson covers Pythagoras' theorem: finding the hypotenuse, finding a shorter side, Pythagorean triples, the converse, problems in context and distance on a coordinate grid.

Starter activity

Draw a right-angled triangle with shorter sides 3 cm and 4 cm. Measure the hypotenuse. Now draw squares on each of the three sides and work out their areas: 9, 16 and 25 square centimetres. What do you notice? Try with 5 cm and 12 cm. Can you say a rule connecting the three areas?

The longest side of a right-angled triangle

Definition: hypotenuse

A **right-angled triangle** has one angle of 90°. The side opposite the right angle is the longest side and is called the **hypotenuse**. The other two sides are the *legs*.

Pythagoras' theorem

Theorem: Pythagoras

$$a^2 + b^2 = c^2$$

where c is the hypotenuse and a and b are the other two sides. In words: **the square on the hypotenuse equals the sum of the squares on the other two sides**. It only works in right-angled triangles. Always sketch the triangle, mark the right angle and label the hypotenuse.

Finding the hypotenuse

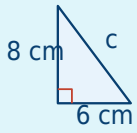
Method: finding the hypotenuse

1. Square the two shorter sides.
2. **Add** them.
3. Take the square root and give the units.

Worked example 1: a perfect square

A right-angled triangle has legs 6 cm and 8 cm. Find the hypotenuse.

$c^2 = 6^2 + 8^2 = 36 + 64 = 100$. $c = \sqrt{100} = 10$ cm.



Not drawn accurately

Worked example 2: a decimal answer

The legs are 4 cm and 7 cm. Find the hypotenuse.

$c^2 = 16 + 49 = 65$. $c = \sqrt{65} = 8.06$ cm to 3 significant figures. Since 65 is not a perfect square the answer is a decimal, so round it sensibly.

Finding a shorter side

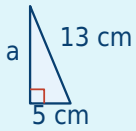
Method: finding a shorter side

1. Rearrange the theorem: $a^2 = c^2 - b^2$.
2. This time **subtract** the squares.
3. Take the square root, and check that the hypotenuse is still the longest side.

Worked example 3: a shorter side

The hypotenuse is 13 cm and one leg is 5 cm. Find the other leg.

$a^2 = 13^2 - 5^2 = 169 - 25 = 144$. $a = \sqrt{144} = 12$ cm.



Not drawn accurately

Worked example 4: a decimal leg

The hypotenuse is 9 cm and one leg is 5 cm. Find the other leg.

$a^2 = 81 - 25 = 56$, so $a = \sqrt{56} = 7.48$ cm to 3 s.f.

Watch out: check the hypotenuse

The hypotenuse must always be the longest side. If your hypotenuse is shorter than one of the legs, you have made a mistake, perhaps adding when you should subtract.

Pythagorean triples and the converse

Definition: Pythagorean triple

A set of three whole numbers that satisfies the theorem is a **Pythagorean triple**. The best-known are 3, 4, 5; 5, 12, 13; 8, 15, 17 and 7, 24, 25. Multiples such as 6, 8, 10 also work.

Key fact: the converse

If the squares of the two shorter sides add up to the square of the longest side, the triangle *is* right-angled. For 8, 15, 17: $64 + 225 = 289 = 17^2$, so yes. For 6, 7, 10: $36 + 49 = 85$, but $10^2 = 100$, so no.

Problems in context

Worked example 5: a ladder

A 5 m ladder leans against a vertical wall with its foot 1.4 m from the wall. How high does it reach?

The ladder is the hypotenuse, so $h^2 = 5^2 - 1.4^2 = 25 - 1.96 = 23.04$, $h = 4.8$ m.

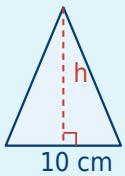


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Worked example 6: an isosceles triangle

An isosceles triangle has base 10 cm and equal sides 13 cm. Find its height and area.

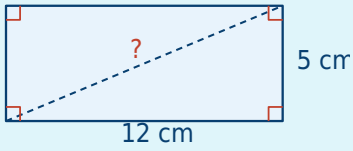
The height cuts the base in half, forming a right-angled triangle with base 5 and hypotenuse 13. $h^2 = 169 - 25 = 144$, $h = 12$ cm. The area is $\frac{1}{2} \times 10 \times 12 = 60\ \text{cm}^2$.



Worked example 7: a rectangle diagonal

A rectangle measures 12 cm by 5 cm. Find the length of its diagonal.

The diagonal is the hypotenuse of a right-angled triangle: $\sqrt{144+25} = \sqrt{169} = 13$ cm.



Distance between two coordinates

Worked example 8: distance on a grid

Find the distance between $(1, 2)$ and $(7, 10)$.

Make a right-angled triangle whose legs are the horizontal and vertical differences: across $= 6$, up $= 8$. Distance $= \sqrt{36+64} = 10$.

Note: a method for every question

Draw a sketch and label the sides a, b and c, with c opposite the right angle. Write the theorem, substitute, work out the squares, add or subtract, then take the square root. Give the units and round only at the end. Many questions hide a right-angled triangle inside a bigger shape: look for the right angle first, then draw the triangle you need.

Watch out: common mistakes

- Adding the squares when you should subtract (when looking for a shorter side).
- Forgetting to take the square root at the end: $c^2 = 100$ does not mean $c = 100$.
- Squaring wrongly: $6^2 = 36$, not 12.
- Using the theorem on a triangle that is not right-angled.
- Forgetting units, or giving too many decimal places.

Worked example 9: a diagonal in a rectangle

A rectangular tablet screen is 24 cm wide and 18 cm high. Find its diagonal.

$d^2 = 24^2 + 18^2 = 576 + 324 = 900$, so $d = 30$ cm. Notice that $18, 24, 30$ is $6 \times (3, 4, 5)$, a multiple of the 3, 4, 5 triple.

Worked example 10: the height of an equilateral triangle

An equilateral triangle has sides of 8 cm. Find its height and area.

The height cuts the base into two halves of 4 cm. $h^2 = 8^2 - 4^2 = 64 - 16 = 48$, so $h = \sqrt{48} = 6.93$ cm. Area $= \frac{1}{2} \times 8 \times 6.93 = 27.7$ cm^2 to 1 decimal place.

Worked example 11: is the triangle right-angled?

A builder measures 1.5 m, 2.0 m and 2.4 m on a corner. Is the corner a right angle?

$1.5^2 + 2.0^2 = 2.25 + 4 = 6.25$. The longest side squared is $2.4^2 = 5.76$. Since $6.25 \neq 5.76$, it is not a right angle. For a true right angle the long side should be 2.5 m.

Worked example 12: finding a side with a ratio

A flagpole is held by a wire 15 m long fixed to the ground 9 m from the base of the pole. Find the height where the wire is attached.

$h^2 = 15^2 - 9^2 = 225 - 81 = 144$, so $h = 12$ m. This is the $3, 4, 5$ triple again, multiplied by 3.

Note: when to add and when to subtract

Ask yourself: am I looking for the longest side? If yes, add the squares and take the square root. If I am looking for a shorter side, I subtract the smaller square from the larger one. A quick check: the hypotenuse must always be the longest side of the triangle, so your answer for it must be bigger than both of the other sides.

Watch out: errors with square roots

- Do not forget the final square root: $c^2 = 100$ means $c = 10$, not 100.
- $\sqrt{a^2 + b^2}$ is not $a + b$: $\sqrt{3^2 + 4^2} = 5$, not 7.
- Give units and say whether you have rounded: an answer such as $\sqrt{65}$ is exact, and 8.06 is rounded.

Did you know?

The theorem is named after Pythagoras, a Greek mathematician from about 530 BC, but Babylonian clay tablets from 1 000 years earlier already list Pythagorean triples. Ancient Egyptian builders are said to have made right angles with a rope knotted into 12 equal parts and pulled into a 3, 4, 5 triangle.

Pythagoras in more complicated problems

Worked example 13: two triangles in one diagram

In a kite-shaped frame, a vertical strut crosses a horizontal strut at right angles. The top part of the frame is made of two sides of 15 cm, and the horizontal strut is 24 cm wide. Find the length of the vertical part above the horizontal strut.

Start by drawing the right-angled triangle you need and label the hypotenuse. The horizontal strut is cut in half by the vertical one, so each half is 12 cm. The sloping side of 15 cm is the hypotenuse, so subtract: $15^2 - 12^2 = 225 - 144 = 81$, and $\sqrt{81} = 9$. The vertical part above the strut is 9 cm. When a diagram contains several triangles, deal with one triangle at a time and write the answer on the diagram before moving to the next one.

Worked example 14: leaving an answer as a surd

Find the exact length of the hypotenuse of a right-angled triangle with shorter sides 2 cm and 3 cm.

Square and add: $2^2 + 3^2 = 4 + 9 = 13$, so the hypotenuse is $\sqrt{13}$ cm. Because 13 is not a square number, $\sqrt{13}$ is irrational, and its decimal never ends. If the question asks for an exact answer, leave it as $\sqrt{13}$ cm. If it asks for a rounded answer, use a calculator: $\sqrt{13} = 3.605\ldots$, which is 3.61 cm to 2 decimal places.

Method: solving a worded Pythagoras problem

1. Draw a clear sketch, even if the question has no diagram, and mark the right angle.
2. Label the lengths you know and use a letter for the length you want.
3. Decide whether the unknown is the hypotenuse (square, add, square root) or a shorter side (square, subtract, square root).
4. Check that all lengths are in the same units before you start.
5. Round only at the end, and write a sentence that answers the question in context, for example "The ladder reaches 4.8 m up the wall."

Note: Pythagoras in the real world

Builders use the 3, 4, 5 triangle to mark out right angles for the corners of houses, using a rope with knots. Satellite navigation and computer games use Pythagoras' theorem millions of times every second to find distances between points.

Otto's tip

Ask “am I finding the longest side?” If yes, add; if no, subtract. Hoot hoot!