Lesson - Probability: sample spaces and Venn diagrams

Year: Year 9

This lesson covers the probability scale, the complement, mutually exclusive events, sample spaces, sets and Venn diagrams, and expected outcomes.

Starter activity

Roll two dice 36 times and tally the totals. Which total do you think is the most common? Now make a table of all 36 ways the dice can land and count how many ways make each total. Does your table agree with your tally? Why might the results from only 36 rolls differ from the table?

Measuring chance

Definition: probability

Probability tells us how likely an event is, on a scale from 0 to 1. An **impossible** event has probability 0, a **certain** event has probability 1, and an event as likely to happen as not has probability $\frac{1}{2}$. Probabilities can be written as fractions, decimals or percentages.

Rule: equally likely outcomes

$$P(\text{event}) = \frac{\text{number of outcomes in the event}}{\text{total number of possible outcomes}}$$

Worked example 1: a spinner

A fair spinner has the numbers 1 to 8. Find $P(\text{even})$ and $P(\text{greater than } 5)$.

$P(\text{even}) = \frac{4}{8} = \frac{1}{2}$. $P(\text{greater than } 5) = \frac{3}{8}$ (the numbers 6, 7 and 8). This is *theoretical* probability: it comes from counting the possible outcomes, not from doing an experiment.

Probabilities add up to 1

Key fact: the complement

The probabilities of **all possible outcomes** of an experiment add up to 1. So $P(\text{event does not happen}) = 1 - P(\text{event happens})$.

Worked example 2: counters and rain

A bag has red, blue and green counters with $P(\text{red}) = 0.3$ and $P(\text{blue}) = 0.45$. Find $P(\text{green})$. The chance of rain is 0.35; find the chance of no rain.

$P(\text{green}) = 1 - 0.3 - 0.45 = 0.25$. The chance of no rain is $1 - 0.35 = 0.65$.

Mutually exclusive events

Definition: mutually exclusive

Two events are **mutually exclusive** if they cannot happen at the same time (rolling a 2 and rolling a 5 on one throw of a die).

Rule: the addition rule

For mutually exclusive events A and B, $P(A \text{ or } B) = P(A) + P(B)$. If $P(A) = 0.3$ and $P(B) = 0.2$, then $P(A \text{ or } B) = 0.5$.

Sample space diagrams

Definition: sample space

When two events happen together, a **sample space** lists every possible outcome systematically in a table or grid. For two fair dice there are $6 \times 6 = 36$ equally likely outcomes.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Worked example 3: two dice

Two dice are added. Find $P(\text{total } 7)$, $P(\text{total} \geq 10)$ and $P(\text{a double})$.

The total 7 appears 6 times, so $P = \frac{6}{36} = \frac{1}{6}$. A total of 10 or more appears 6 times (10 three times, 11 twice, 12 once), so the probability is also $\frac{1}{6}$. A double has probability $\frac{6}{36} = \frac{1}{6}$.

Worked example 4: coins and a spinner

A coin and a spinner numbered 1, 2, 3 are used together. Find $P(\text{head and } 3)$. For two coins find $P(\text{at least one head})$.

The outcomes are H1, H2, H3, T1, T2, T3: six in total, so $P(\text{head and } 3) = \frac{1}{6}$. Two coins give HH, HT, TH, TT, so $P(\text{at least one head}) = \frac{3}{4}$.

Sets and Venn diagrams

Definition: sets and Venn notation

A **set** is a collection of items, written in curly brackets: $\{2, 4, 6\}$. A Venn diagram shows sets as overlapping circles inside a rectangle that stands for the **universal set** ξ.

- $A \cap B$ is the **intersection**: items in both A and B (the overlap).
- $A \cup B$ is the **union**: items in A or B or both.
- A' is the **complement**: items not in A.
- $n(A)$ is the number of items in A.

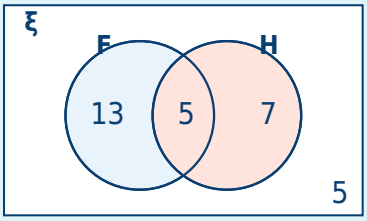
Method: filling in a Venn diagram

1. Fill in the overlap first.
2. Subtract the overlap from each set total to get the "only" regions.
3. Add all the regions and subtract from the total to find the number in neither set.

Worked example 5: football and hockey

In a class of 30, 18 play football (F), 12 play hockey (H) and 5 play both. Find the probabilities of neither, F or H, both, and football only.

Football only: $18 - 5 = 13$. Hockey only: $12 - 5 = 7$. In at least one: $13 + 5 + 7 = 25$. Neither: $30 - 25 = 5$. So $P(\text{neither}) = \frac{5}{30} = \frac{1}{6}$, $P(F \cup H) = \frac{25}{30} = \frac{5}{6}$, $P(F \cap H) = \frac{5}{30} = \frac{1}{6}$ and $P(\text{football only}) = \frac{13}{30}$.



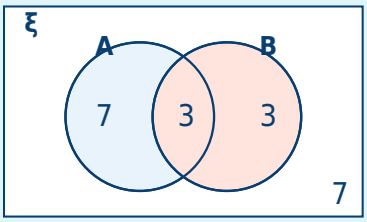
Rule: counting a union

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$. The overlap is counted twice when you add, so subtract it once.

Worked example 6: numbers 1 to 20

From the numbers 1 to 20, A is the even numbers and B the multiples of 3. Find $n(A \cup B)$ and the probability that a random number is in A or B.

$n(A) = 10$, $n(B) = 6$ (3, 6, 9, 12, 15, 18) and $A \cap B = \{6, 12, 18\}$, so $n(A \cap B) = 3$. Then $n(A \cup B) = 10 + 6 - 3 = 13$, and the probability is $\frac{13}{20}$.



Using probability to predict

Rule: expected number

The **expected number** of times an event happens is $\text{probability} \times \text{number of trials}$. If $P(\text{red}) = \frac{1}{4}$ and a spinner is spun 60 times, you expect $\frac{1}{4} \times 60 = 15$ reds. The real result will usually be close but not exactly equal.

Watch out: common mistakes

- Giving a probability bigger than 1 or a negative one.
- Counting outcomes as equally likely when they are not (the total of two dice is not equally likely: 7 is more likely than 2).
- Forgetting to subtract the overlap when counting a union, or putting the total of a set into the overlap region.
- Forgetting the items outside both circles.
- Not simplifying the fraction when asked.

Worked example 7: a two-way table

In a group of 50 pupils, 28 are girls. 15 of the girls and 9 of the boys have a pet. Find the probability that a pupil chosen at random is a boy without a pet.

Boys: $50 - 28 = 22$. Boys with a pet: 9. Boys without a pet: $22 - 9 = 13$. So the

probability is $\frac{13}{50} = 0.26$.

Worked example 8: expected number

The probability that a train is late is 0.08. How many late trains are expected in 250 journeys?

Expected number $= 250 \times 0.08 = 20$ late trains. This is an estimate for the long run, not a promise: the actual number could be a little higher or lower.

Worked example 9: the union with the addition rule

$P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.2$. Find $P(A \cup B)$ and $P(\text{neither})$.

$P(A \cup B) = 0.5 + 0.4 - 0.2 = 0.7$. The 0.2 must be subtracted because it is counted in both $P(A)$ and $P(B)$. $P(\text{neither}) = 1 - 0.7 = 0.3$.

Worked example 10: choosing between two outcomes

A bag has 3 red, 4 blue and 5 green counters. Find the probability of taking a blue or green counter.

There are 12 counters. Blue or green: $4 + 5 = 9$. So $P = \frac{9}{12} = \frac{3}{4}$. Another way: $1 - P(\text{red}) = 1 - \frac{3}{12} = \frac{3}{4}$. Using both methods is a good check.

Note: write probabilities in the form you are asked for

A probability can be a fraction, a decimal or a percentage, but never greater than 1 and never negative. If a question gives decimals, answer in decimals. Simplify fractions. If you are asked for the expected number of times, multiply the probability by the number of trials.

Watch out: common mistakes

- "Even chance" does not mean two outcomes always have probability $\frac{1}{2}$: the outcomes must be *equally likely*.
- When two events can happen together, do not just add their probabilities; subtract the overlap.
- In a sample space, list outcomes in order so that you do not miss any: for two dice there must be $6 \times 6 = 36$.

Did you know?

Probability began with a gambling puzzle. In 1654 the French mathematicians Blaise

Pascal and Pierre de Fermat exchanged letters about how to share the prize in an unfinished game of chance. Their letters are often said to be the start of probability theory, which today guides insurance, weather forecasts and medical tests.

Theory and experiment

Definition: relative frequency

When outcomes are not equally likely, or when we do not know the probabilities, we can carry out an experiment. The **relative frequency** of an outcome is the number of times it happens divided by the number of trials. It is an estimate of the probability, and the estimate usually gets better as the number of trials increases.

Worked example 11: is the dice biased?

Leo rolls a dice 300 times and gets a six 78 times. Do you think the dice is fair?

The relative frequency of a six is $78 \div 300 = 0.26$. For a fair dice the theoretical probability is $\frac{1}{6}$, which is about 0.17, so we would expect about $300 \times \frac{1}{6} = 50$ sixes.

Getting 78 sixes is a lot more than 50, and 300 is a large number of trials, so the result suggests the dice is biased towards six. Leo cannot be completely sure, but he has good evidence. With only 12 rolls, a few extra sixes would not tell us very much.

Method: choosing how to list outcomes

1. For one event, list the outcomes in a simple list or table of frequencies.
2. For two events, such as two dice or a coin and a spinner, draw a sample space grid with one event across the top and the other down the side.
3. For groups of people with two properties, such as sports or languages, use a Venn diagram and always fill in the overlap first.
4. Count the outcomes you want and divide by the total, simplifying the fraction at the end.

Otto's tip

In Venn diagram problems, always fill in the middle (the overlap) first, then work outwards. Check that every region adds to the total. Hoot hoot!