Lesson - Number: approximation, error intervals and calculators

Year: Year 9

This lesson covers rounding, estimating, error intervals, bounds in calculations, calculator use and types of number.

Starter activity

Measure the width of your desk with a ruler marked in millimetres, then with one marked only in centimetres. Write both answers. Which is more accurate? A friend says the desk is 80 cm wide to the nearest cm. What is the shortest and longest it could really be? Draw a number line to show the possible values.

Why we round and estimate

Definition: rounding and estimating

Rounding replaces a number with a simpler one that is close to it. **Estimating** uses rounded numbers to get a quick answer so that you can check whether a calculator answer is sensible. A builder measures to the nearest centimetre; a runner is timed to the nearest hundredth of a second.

Rounding to decimal places and significant figures

Rule: decimal places and significant figures

To round to a given number of **decimal places (d.p.)**, look at the digit just after the place you want. If it is 5 or more, round up; if it is 4 or less, leave the digit alone.

For **significant figures (s.f.)**, the first significant figure is the first non-zero digit from the left. Count from there. Zeros at the start of a decimal such as 0.0047 are not significant, but you must keep them as place holders.

Worked example 1: significant figures

Round 0.0047382 to 3 s.f. and 35 678 to 2 s.f.

0.0047382 to 3 s.f. is 0.00474. 35 678 to 2 s.f. is 36 000 (the zeros keep the place value).

Estimating

Method: estimating a calculation

1. Round every number to **1 significant figure**.
2. Calculate with the rounded numbers.
3. Compare with the calculator answer to check it is sensible.

Worked example 2: an estimate

Estimate $\dfrac{19.8 \times 4.1}{0.52}$.

Round to 1 s.f.: $\dfrac{20 \times 4}{0.5} = 160$ (dividing by 0.5 is the same as doubling). The exact answer is 156.1 to 4 s.f., so the estimate shows the calculator answer is sensible.

Error intervals

Definition: error interval

When a number has been rounded, the original value could have been anywhere in a small range. This range is written as an **error interval**:

$$\text{lower bound} \leq x < \text{upper bound}$$

Rule: finding the bounds

Take **half of the rounding unit**, subtract it for the lower bound and add it for the upper bound. The lower bound is included ($\leq$) but the upper bound is not (), because a value exactly on the upper bound would have been rounded up to the next number. If a number is **truncated**, the interval starts at the number itself and goes up by a whole unit: 5.3 truncated to 1 d.p. satisfies $5.3 \leq x < 5.4$.

Worked example 3: error intervals

Write error intervals for (a) 7 cm to the nearest cm, (b) 250 g to the nearest 10 g, (c) 3.4 to 1 d.p.

(a) The rounding unit is 1, half is 0.5, so $6.5 \leq x < 7.5$. (b) Half of 10 is 5, so $245 \leq m < 255$. (c) The rounding unit is 0.1, half is 0.05, so $3.35 \leq x < 3.45$.

Bounds in calculations

Key fact: combining bounds

For the largest possible answer combine upper bounds when adding and multiplying; for the smallest combine lower bounds. For subtraction and division be careful: the

largest value of $a - b$ uses the upper bound of a and the *lower* bound of b.

Worked example 4: area of a rectangle

A rectangle has sides 8 cm and 5 cm, each to the nearest cm. Find the greatest and least possible area.

Greatest: $8.5 \times 5.5 = 46.75\ \text{cm}^2$. Least: $7.5 \times 4.5 = 33.75\ \text{cm}^2$.

Using a calculator accurately

Worked example 5: fractions on a calculator

Work out $\dfrac{4.7 + 3.2}{5.1 - 2.9}$ to 3 s.f.

A calculator follows BIDMAS but only does what you type, so use brackets or the fraction key: type (4.7 + 3.2) ÷ (5.1 − 2.9). The answer is 3.5909..., which is 3.59 to 3 s.f. Do not round in the middle of a calculation; keep the full display or use the answer memory and round only at the end.

Different kinds of number

Definition: integer, rational, irrational and real numbers

The **integers** are ..., −2, −1, 0, 1, 2, ... A **rational** number can be written as a fraction $\dfrac{a}{b}$ of two integers, so terminating decimals such as 0.25 and recurring decimals such as 0.333... are rational. **Irrational** numbers, such as π and $\sqrt{2}$, have decimals that never end and never repeat. Together they make up the **real** numbers.

Watch out: common mistakes

- Writing the upper bound with $\leq$. It must be .
- Rounding to decimal places when the question asks for significant figures.
- Forgetting the zero place holders: 35 678 to 2 s.f. is 36 000, not 36.
- Rounding too early in a multi-step calculation.
- Rounding 0.8 to 1 d.p. to get the interval $0.7 \leq x < 0.9$. The rounding unit is 0.1, so the interval is $0.75 \leq x < 0.85$.

Worked example 6: money and error intervals

A charity says it raised £4 300, correct to 2 significant figures. Write the error interval.

The second significant figure is the hundreds digit, so the accuracy is £100. Half of

100 is 50. The error interval is $4250 \leq x$. The charity could have raised as little as £4 250 and as much as £4 349.99.

Worked example 7: bounds of a speed

A car travels 150 km, to the nearest 10 km, in 2.0 hours, to 1 decimal place. Find the greatest possible average speed.

Distance: $145 \leq d$. Time: $1.95 \leq t$. For the greatest speed use the greatest distance and the least time: $155 \div 1.95 = 79.5$ km/h (1 decimal place).

Worked example 8: choosing the accuracy of an answer

A rectangle is 6.3 cm by 4.8 cm, each to 1 decimal place. The area is 30.24 cm^2 on a calculator. How accurately should it be given?

Lower bound: $6.25 \times 4.75 = 29.6875$. Upper bound: $6.35 \times 4.85 = 30.7975$. To 1 significant figure both are 30. So the area is 30 cm^2 to 1 significant figure. Giving 30.24 would suggest an accuracy that the measurements cannot support.

Worked example 9: converting a recurring decimal

Show that $0.\dot{4}\dot{5} = \frac{5}{11}$.

Let $x = 0.454545\ldots$ Then $100x = 45.4545\ldots$ Subtract: $99x = 45$, so $x = \frac{45}{99} = \frac{5}{11}$.

Because it can be written as a fraction, it is rational.

Note: rounding the final answer only

When a calculation has several steps, keep the full calculator value for as long as you can, and round only the final answer. If you round in the middle you can build up errors. For example, $\sqrt{2} = 1.41$ gives $1.41^2 = 1.9881$, which is not 2. Write down enough figures to be safe, usually one more than you need.

Watch out: bounds in division

- For the greatest answer to $a \div b$ use the upper bound of a and the *lower* bound of b.
- For the least answer to $a - b$ use the lower bound of a and the *upper* bound of b.
- The upper bound is not a value that is included, so use , not $\leq$, with the upper bound.

Did you know?

Computers store most decimals as rounded approximations. A famous error in 1991

caused a Patriot missile system to miss its target, because a tiny rounding error in its clock built up over 100 hours. The lesson: small rounding errors can grow, so engineers always check their bounds.

Using a calculator sensibly

Method: checking a calculator answer

1. Before you press any keys, write down a rough estimate by rounding each number to 1 significant figure.
2. Type the calculation in carefully, using brackets around the whole top and the whole bottom of a fraction, or use the fraction key.
3. Write down the full calculator display before you round, so that you can show your working.
4. Compare the display with your estimate. If they are very different, for example 5 and 500, you have probably pressed a wrong key or left out a bracket.
5. Round the final answer to the accuracy asked for in the question, and include the units.

Worked example 10: an estimate catches a mistake

Ruby works out $\dfrac{48.6 + 21.3}{6.8}$ and writes 51.7. Is she right?

Round each number to 1 significant figure: $\dfrac{50 + 20}{7} = \dfrac{70}{7} = 10$. Ruby's answer is about five times too big, so something has gone wrong. She probably typed $48.6 + 21.3 \div 6.8$, which the calculator works out as $48.6 + 3.13\ldots = 51.7$ because division is done before addition. The correct calculation is $69.9 \div 6.8 = 10.279\ldots$, which is 10.3 to 3 significant figures and agrees with the estimate.

Note: rounding in everyday life

Rounding is everywhere. Newspapers give crowd sizes to the nearest hundred or thousand, weather forecasts round temperatures to the nearest degree, and shops sometimes round prices. Engineers use bounds to make sure that parts will always fit together, even when each part is a little bigger or smaller than its stated size.

Otto's tip

Always ask “what is the rounding unit?” first. Halve it, then add and subtract. Then write the lower bound first with $\leq$ and the upper bound with . Hoot hoot!