Lesson - Expanding binomials and taking out common factors

Year: Year 9

This lesson covers equivalent expressions, expanding single and double brackets, squaring a bracket, the difference of two squares, three brackets and taking out common factors.

Starter activity

Draw a rectangle that is 13 cm by 12 cm on squared paper and split it into $10 + 3$ and $10 + 2$. Find the four smaller areas 10×10, 10×2, 3×10 and 3×2 and add them up. Does it match $13 \times 12 = 156$? Now replace the 10 by x. Which four areas do you get? What does this tell you about $(x + 3)(x + 2)$?

Expressions that stay equivalent

Definition: equivalent expressions

Two expressions are **equivalent** if they have the same value for every value of the letters. **Expanding** removes brackets and **factorising** puts them back: they are opposite processes.

Expanding a single bracket

Rule: expanding a bracket

Multiply every term inside the bracket by the term outside. Be careful with negative signs.

Worked example 1: single brackets

Expand (a) $3(x + 4)$ and (b) $2(x + 3) - 3(x - 1)$.

(a) $3x + 12$. (b) $2x + 6 - 3x + 3 = -x + 9$. The minus sign in front of $3(x - 1)$ changes the sign of both terms inside.

Expanding two brackets (binomials)

Definition: binomial

A **binomial** is an expression with two terms, such as $x + 3$. To expand $(x + 2)(x + 3)$ every term in the first bracket must multiply every term in the second.

Method: the grid (or FOIL)

1. Draw a grid with the terms of one bracket across the top and the other down the side.
2. Multiply to fill each box (First, Outside, Inside, Last).
3. Add all four boxes and collect like terms.

$\times$	x	$+3$
x	x^2	$3x$
$+2$	$2x$	6

$x^2 + 3x + 2x + 6 = x^2 + 5x + 6$.

Worked example 2: double brackets

Expand (a) $(x+7)(x-2)$, (b) $(x-4)(x-3)$ and (c) $(2x+1)(x+5)$.

(a) $x^2 - 2x + 7x - 14 = x^2 + 5x - 14$. (b) $x^2 - 3x - 4x + 12 = x^2 - 7x + 12$; a negative times a negative is positive. (c) $2x^2 + 10x + x + 5 = 2x^2 + 11x + 5$.

Squaring a bracket and the difference of two squares

Watch out: squaring a bracket

$(x+5)^2$ means $(x+5)(x+5)$. It does *not* equal $x^2 + 25$.

$$(x+5)^2 = x^2 + 5x + 5x + 25 = x^2 + 10x + 25$$

The middle term comes from two lots of $5x$. Similarly $(x-3)^2 = x^2 - 6x + 9$ and $(2x+3)^2 = 4x^2 + 12x + 9$.

Rule: difference of two squares

When the brackets differ only by the sign, the middle terms cancel:

$$(a+b)(a-b) = a^2 - b^2$$

For example $(x+4)(x-4) = x^2 - 4x + 4x - 16 = x^2 - 16$.

Worked example 3: three brackets

Expand $(x+2)(x-1)(x+3)$.

Expand two brackets first: $(x+2)(x-1) = x^2 + x - 2$. Then $(x^2 + x - 2)(x+3) = x^3 + 3x^2 + x^2 + 3x - 2x - 6 = x^3 + 4x^2 + x - 6$. Check with $x = 1$: the original gives $3 \times 0 \times 4 = 0$ and the answer gives $1 + 4 + 1 - 6 = 0$.

Taking out common factors

Definition: highest common factor

To factorise an expression, find the **highest common factor (HCF)** of all the terms: the largest number and the highest power of each letter that divides every term. Write it outside the bracket and put what is left inside.

Worked example 4: common factors

Factorise (a) $6x + 9$, (b) $x^2 + 5x$ and (c) $12x^2y - 18xy^2$.

(a) The HCF of 6 and 9 is 3: $3(2x + 3)$. (b) Both terms have a factor of x: $x(x + 5)$. (c) The HCF of 12 and 18 is 6, the lowest power of x is x and of y is y, so the HCF is $6xy$: $6xy(2x - 3y)$.

Key fact: checking factorising

Always check by expanding your answer. Also check that the bracket has no common factor left: $2(3x + 6)$ is not fully factorised, because $3x + 6$ still has a factor of 3.

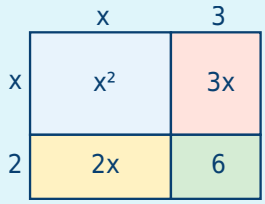
Watch out: common mistakes

- Writing $(x + 5)^2 = x^2 + 25$ and forgetting the middle term.
- Sign errors when multiplying negatives: $-3 \times -4 = +12$.
- Only taking out part of the HCF, such as writing $6x + 12 = 2(3x + 6)$.
- Forgetting the 1 left behind: $4a^2b + 4a = 4a(ab + 1)$, not $4a(ab)$.
- Forgetting to collect like terms at the end.

Worked example 5: the area model

Use a grid to expand $(x + 3)(x + 2)$.

Draw a rectangle with sides $x + 3$ and $x + 2$ and split it into four parts: $x \times x = x^2$, $x \times 2 = 2x$, $3 \times x = 3x$ and $3 \times 2 = 6$. Add: $x^2 + 2x + 3x + 6 = x^2 + 5x + 6$. When $x = 10$ this gives $100 + 50 + 6 = 156$, which equals 13×12, so the expansion is correct.



Worked example 6: factorising with two letters

Factorise fully $12x^2y + 18xy^2$.

Find the HCF of the numbers: 6. Find the HCF of the letters: x appears at least once and y at least once, so take out xy. The HCF is $6xy$. Divide each term: $12x^2y \div 6xy = 2x$ and $18xy^2 \div 6xy = 3y$. So the answer is $6xy(2x + 3y)$. Expand it to check: $12x^2y + 18xy^2$.

Worked example 7: expand, simplify, then substitute

Show that $(x + 5)^2 - (x - 5)^2 = 20x$ and use it to find $105^2 - 95^2$.

$(x + 5)^2 = x^2 + 10x + 25$ and $(x - 5)^2 = x^2 - 10x + 25$. Subtracting gives $20x$. With $x = 100$: $105^2 - 95^2 = 20 \times 100 = 2000$. Check: $11025 - 9025 = 2000$.

Worked example 8: expanding a cubic

Expand $x(x + 2)(x + 3)$.

Expand the double bracket first: $(x + 2)(x + 3) = x^2 + 5x + 6$. Then multiply by x: $x^3 + 5x^2 + 6x$. This is the volume of a cuboid with sides x, $x + 2$ and $x + 3$.

Worked example 9: expressions for a perimeter

A rectangle has sides $(3x + 2)$ cm and $(x + 5)$ cm. Write the perimeter and the area, both in simplest form.

Perimeter: $2(3x + 2) + 2(x + 5) = 6x + 4 + 2x + 10 = 8x + 14$. Area: $(3x + 2)(x + 5) = 3x^2 + 15x + 2x + 10 = 3x^2 + 17x + 10$. When $x = 2$ the sides are 8 cm and 7 cm, so the perimeter is 30 cm and the area is 56 cm^2. Check: $8 \times 2 + 14 = 30$ and $12 + 34 + 10 = 56$.

Method: factorising fully

1. Find the highest common factor of the numbers.
2. Find any letter that appears in every term, using its smallest power.
3. Write the HCF outside the bracket.
4. Divide each term by the HCF and write the results inside the bracket.
5. Expand to check.

Note: factorising and expanding are inverses

To expand, you multiply to remove brackets. To factorise, you divide out the common factor and put brackets back. If you are not sure that you have factorised fully, ask: do the terms inside the bracket still share a common factor? If so, take it out as well. Expanding your answer is the best check.

Watch out: signs and squares

- $(x+3)^2$ is x^2+6x+9, not x^2+9. Always write the bracket twice.
- A minus sign in front of a bracket changes the sign of every term inside it: $-(x-4)=-x+4$.
- When you factorise $6x+9$ as $3(2x+3)$, do not forget the 3 inside the bracket: $3(2x+3)$ and $3\times(2x)$ are different things.

Did you know?

The word algebra comes from the Arabic al-jabr, in the title of a book written by the mathematician al-Khwarizmi around the year 820. The method of expanding brackets by splitting a rectangle into smaller areas is almost as old: Greek mathematicians used it to prove results about squares more than 2 000 years ago.

Putting it all together

Worked example 10: a negative outside a double bracket

Expand and simplify $5x-(x-3)(x+2)$.

Do not try to deal with the minus sign and the two brackets at the same time. First expand the double bracket on its own, keeping it inside a bracket: $(x-3)(x+2)=x^2+2x-3x-6=x^2-x-6$. Now write $5x-(x^2-x-6)$. The minus sign in front of the bracket changes the sign of every term inside it, so the expression becomes $5x-x^2+x+6$. Finally collect the like terms to get $-x^2+6x+6$. A quick check with $x=1$ gives $5-(-2)(3)=11$ for the original and $-1+6+6=11$ for the answer, so the working is correct.

Method: choosing between expanding and factorising

1. Read the instruction carefully. "Expand" or "multiply out" means remove the brackets; "factorise" means put brackets in.
2. If you must expand, decide how many brackets there are. One bracket needs one multiplication per term; two brackets need four products; three brackets are done two at a time.
3. If you must factorise, look first for a common factor in every term. Then look for a difference of two squares, which has exactly two square terms separated by a minus sign.
4. Write a full line of working before you simplify, so that a teacher can follow every step and you can spot your own slips.
5. Finish with a check: substitute a small number such as 2 into the question and into your answer. If the two values are different, look back for a sign error or a missing term.

Worked example 11: factorising in context

A path around a square pond has an area of $8x + 16$ square metres. Factorise the expression and explain what the factorised form tells you.

The highest common factor of 8 and 16 is 8, and the letter x does not appear in the second term, so the HCF is just 8. Dividing each term by 8 leaves x and 2, so $8x + 16 = 8(x + 2)$. The factorised form shows that the path can be split into 8 equal strips, each with an area of $x + 2$ square metres. It also makes calculations easier: when $x = 3$, the area is $8 \times 5 = 40$ square metres, which is quicker than working out $24 + 16$.

Note: why algebra skills matter

Expanding and factorising are the tools that you will use to solve quadratic equations in Year 10 and Year 11. A quadratic such as $x^2 + 5x + 6 = 0$ is solved by factorising it into two brackets, and many GCSE questions ask you to "show that" an area or a volume equals a given expression, which means expanding carefully and setting out each line. The more fluent you become now, the easier those topics will feel later, so practise little and often rather than all at once.

Watch out: writing the final answer

Examiners expect the answer in its simplest form. Always put the terms in order, starting with the highest power of the letter, then the term in x, then the number. Do not leave two terms that could be collected, and do not leave a bracket that could still be factorised further. If a question says "hence", use the answer you have just found rather than starting again from the beginning.

Otto's tip

Use the grid for every double bracket until you never make a slip, and always check an answer by substituting a number such as $x = 2$ into both sides. Hoot hoot!