Lesson - Direct and inverse proportion

Year: Year 9

This lesson covers direct and inverse proportion, finding the constant of proportionality, graphs and how to decide which type a situation is.

Starter activity

Share a bag of 24 sweets equally. With 2 people each gets 12. With 3 people each gets 8. With 4 people each gets 6. Now look at the cost: 1 bag costs £1.20, 2 bags cost £2.40, 3 bags cost £3.60. In which list does more people mean less each, and in which does more bags mean more money? Write a rule for each.

Two ways quantities can be linked

Note: why proportion matters

If you buy more pens the total cost goes up, and if more workers share a job it takes less time. Proportion is the language we use to describe these links precisely. There are two main types: direct proportion and inverse proportion.

Direct proportion

Definition: direct proportion

Two quantities are in **direct proportion** if when one doubles, the other doubles; when one is trebled, the other is trebled; and so on. The ratio $\frac{y}{x}$ always stays the same. We write $y \propto x$ (read “y is proportional to x”).

Rule: direct proportion

$$y = kx$$

where k is the **constant of proportionality**. The graph of $y = kx$ is a **straight line through the origin** with gradient k.

Method: solving a proportion problem

1. Decide whether the situation is direct or inverse by asking what happens when one quantity doubles.
2. Write the equation with k and substitute the pair of values you are given to find k.
3. Rewrite the equation with the value of k in it.
4. Substitute the new value and solve.

5. Check: in direct proportion a bigger x must give a bigger y; in inverse proportion a bigger x must give a smaller y.

Worked example 1: finding k

y is directly proportional to x, and $y = 20$ when $x = 4$. Find y when $x = 9$.

$y = kx$, so $20 = k \times 4$, giving $k = 5$. The equation is $y = 5x$. When $x = 9$, $y = 5 \times 9 = 45$.

Worked example 2: the unitary method

5 pens cost £3.50. Find the cost of 8 pens.

The cost of one pen is $3.50 \div 5 = £0.70$, so 8 pens cost $8 \times 0.70 = £5.60$. This is the unitary method, which is the same idea as finding k.

Inverse proportion

Definition: inverse proportion

Two quantities are in **inverse proportion** if when one doubles, the other halves. The *product* xy stays the same. We say that y is inversely proportional to x, which is equivalent to saying that y is proportional to $\dfrac{1}{x}$.

Rule: inverse proportion

$$y \propto \frac{1}{x} \quad \Longleftrightarrow \quad y = \frac{k}{x}$$

The graph is a **reciprocal curve** that gets close to both axes but never touches them.

Worked example 3: inverse proportion

y is inversely proportional to x, and $y = 6$ when $x = 4$. Find y when $x = 8$.

$y = \dfrac{k}{x}$, so $6 = \dfrac{k}{4}$ and $k = 24$. The equation is $y = \dfrac{24}{x}$. When $x = 8$, $y = \dfrac{24}{8} = 3$. Doubling x from 4 to 8 halved y from 6 to 3, as expected.

Worked example 4: workers and days

It takes 6 workers 10 days to build a wall. How long would 4 workers take?

Total work is $6 \times 10 = 60$ worker-days. With 4 workers: $60 \div 4 = 15$ days. More workers means fewer days, so this is inverse proportion (assuming everyone works at the same rate).

How to decide which type

Key fact: direct or inverse?

	Direct	Inverse
Behaviour	both increase together	one increases, the other decreases
Test	$y \div x$ is constant	$x \times y$ is constant
Equation	$y = kx$	$y = \frac{k}{x}$
Graph	straight line through (0,0)	reciprocal curve

Worked example 5: reading a table

x: 1, 2, 4, 8 and y: 16, 8, 4, 2. Which type of proportion is this?

The products xy are $16, 16, 16, 16$, so this is inverse proportion with $k = 16$. The equation is $y = \frac{16}{x}$.

Watch out: not every line is proportion

The line $y = 3x + 2$ does not pass through the origin (when $x = 0$, $y = 2$), so y is **not** directly proportional to x. And $y = x + 5$ is neither direct nor inverse.

Interpreting equations

Note: recognising proportion in equations

- $y = 7x$: direct proportion (constant 7).
- $y = \frac{20}{x}$: inverse proportion (constant 20).
- Speed and time over a fixed distance: $t = \frac{d}{v}$, inverse proportion.
- Cost and quantity at a fixed price: $C = pn$, direct proportion.
- Currency exchange: with £1 = 1.15 euros, $E = 1.15P$ is direct proportion.

Watch out: common mistakes

- Using direct proportion when the situation is inverse (for example, dividing 60 by 4 workers but then multiplying).
- Forgetting to find k first.
- Thinking that every straight line shows direct proportion: it must go through the origin.
- Mixing up the formulae: for inverse proportion multiply x and y; for direct

proportion divide them.

Worked example 6: an equation from one pair

The cost C of a taxi ride is directly proportional to the distance d in miles. A 7-mile ride costs £14. Find a formula and the cost of an 11-mile ride.

$C = kd$. Substituting, $14 = 7k$, so $k = 2$ and $C = 2d$. For 11 miles: $C = 2 \times 11 = £22$.

Worked example 7: testing a table

Is y in direct or inverse proportion to x in this table?

Table: $x = 1, 2, 4, 8$ and $y = 24, 12, 6, 3$. Check the products: $1 \times 24 = 24$, $2 \times 12 = 24$, $4 \times 6 = 24$, $8 \times 3 = 24$. The product is constant, so y is inversely proportional to x, with $y = \dfrac{24}{x}$.

Worked example 8: a mixed problem

8 machines make 600 toys in 5 hours. How long would 10 machines take to make 900 toys?

First find the rate of one machine: $600 \div 8 \div 5 = 15$ toys per hour. Ten machines make $10 \times 15 = 150$ toys per hour. So 900 toys take $900 \div 150 = 6$ hours.

Worked example 9: reading a conversion

A recipe for 4 people needs 300 g of rice. How much rice is needed for 10 people, and how many people can be fed with 1.5 kg?

Rice per person $= 300 \div 4 = 75$ g. For 10 people: $10 \times 75 = 750$ g. For 1.5 kg $= 1500$ g: $1500 \div 75 = 20$ people. Both answers use the same constant, 75 g per person, which is the constant of proportionality.

Note: how to spot proportion in a graph

A direct proportion graph is a straight line through the origin $(0, 0)$. An inverse proportion graph is a curve: as x gets bigger, y gets smaller, but neither ever reaches zero. A straight line that does not start at the origin is *not* direct proportion, even if it goes up. The cost of a taxi with a fixed starting charge is a good example.

Watch out: dividing or multiplying?

- For direct proportion, *divide* to find $k = y \div x$. For inverse proportion, *multiply* to find $k = xy$.
- Always check your answer is sensible: in inverse proportion, more of one thing must give less of the other.

- Round up when you need a whole number of people or items, such as 4.8 painters needing 5.

Did you know?

Direct proportion is the idea behind every map scale and every recipe. Inverse proportion appears in physics: if you halve the distance to a loudspeaker the sound is four times as loud, because the loudness is inversely proportional to the square of the distance.

Choosing the right method

Worked example 10: best buys

A 400 g jar of honey costs £3.20 and a 650 g jar costs £4.94. Which jar is better value?

The price of honey is directly proportional to its mass within each jar, so we can compare the cost of the same amount. Find the cost of 1 g in each jar. For the small jar, $3.20 \div 400 = 0.008$ pounds, which is 0.8p per gram. For the large jar, $4.94 \div 650 = 0.0076$ pounds, which is 0.76p per gram. The large jar is cheaper per gram, so it is the better buy. You could also compare the mass you get for £1: $400 \div 3.20 = 125$ g and $650 \div 4.94 \approx 131.6$ g, which gives the same conclusion.

Worked example 11: sharing the work

It takes 4 identical taps 9 minutes to fill a paddling pool. How long would 6 taps take?

More taps fill the pool faster, so the time goes down when the number of taps goes up: this is inverse proportion. The total amount of work is always the same. Four taps for 9 minutes is $4 \times 9 = 36$ tap-minutes. Share this between 6 taps: $36 \div 6 = 6$ minutes. A sensible check is that the answer must be less than 9 minutes, which it is.

Method: deciding which type of proportion

1. Ask yourself: if one quantity doubles, what happens to the other?
2. If the other quantity also doubles, the relationship is direct. Divide to find the constant, $k = y \div x$.
3. If the other quantity halves, the relationship is inverse. Multiply to find the constant, $k = x \times y$.
4. If neither happens, the quantities are not proportional, and you should look for another kind of rule, such as a fixed charge plus a price per item.
5. Use the constant to answer the question, then check that the answer makes sense in the real situation.

Note: proportion in science

Proportion appears in many science lessons. The extension of a spring is directly proportional to the force pulling it, as long as the spring is not stretched too far; this is called Hooke's law. The pressure of a gas kept at the same temperature is inversely proportional to its volume, which is Boyle's law. Knowing how to find and use a constant of proportionality in maths will help you in physics and chemistry as well.

Otto's tip

Ask “if one doubles, what happens to the other?” before writing anything. Doubles: direct. Halves: inverse. Hoot hoot!