Lesson - Angles in polygons

Year: Year 9

This lesson covers polygons, angle facts, the sum of interior angles, interior and exterior angles of regular polygons and using polygon facts in problems.

Starter activity

Walk around the edge of a large square marked on the floor with chalk. At each corner you turn left. How much do you turn at each corner? How much do you turn altogether when you are back where you started facing the same way? Now try a triangle and a hexagon. What do you notice about the total turn?

Polygons and their angles

Definition: polygon, regular and irregular

A **polygon** is a closed 2D shape made of straight sides. A polygon is **regular** if all its sides are equal and all its angles are equal. If it is not regular, it is **irregular**.

Sides	3	4	5	6	7	8	9	10	12
Name	triangle	quadrilateral	pentagon	hexagon	heptagon	octagon	nonagon	decagon	dodecagon

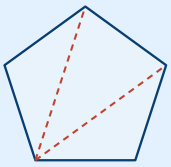
Key fact: angles you already know

- Angles in a triangle add up to 180°.
- Angles in a quadrilateral add up to 360°.
- Angles on a straight line add up to 180°.
- Angles around a point add up to 360°.

Sum of the interior angles

Definition: interior angles

The **interior angles** are the angles inside the polygon. To find their sum, pick one vertex (corner) and draw diagonals from it to every other vertex that is not next to it. This cuts an n-sided polygon into $n - 2$ triangles, and each triangle has angles summing to 180°.



A pentagon splits into 3 tria

Rule: sum of interior angles

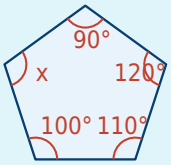
$$\text{sum of interior angles} = (n-2) \times 180^\circ$$

Polygon	n	Triangles	Sum
Triangle	3	1	180°
Quadrilateral	4	2	360°
Pentagon	5	3	540°
Hexagon	6	4	720°
Octagon	8	6	1080°

Worked example 1: a missing angle

Find the missing angle x in a pentagon with angles 100°, 110°, 120°, 90° and x.

The sum is 540°. The four known angles add to 420°. So $x = 540 - 420 = 120^\circ$.



Not drawn accurately

Worked example 2: finding the number of sides

The interior angles of a polygon add up to 1440°. How many sides does it have?

$(n-2) \times 180 = 1440$. Divide by 180: $n - 2 = 8$. So $n = 10$, a decagon.

Interior angle of a regular polygon

Rule: interior angle of a regular polygon

All the angles in a regular polygon are equal, so divide the sum by the number of sides:

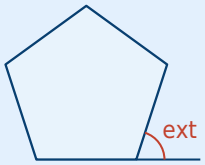
$$\text{interior angle} = \frac{(n-2)\times 180^\circ}{n}$$

A regular hexagon: $\frac{720}{6} = 120^\circ$. A regular octagon: $\frac{1080}{8} = 135^\circ$. A regular pentagon: $\frac{540}{5} = 108^\circ$.

Exterior angles

Definition: exterior angle

If you extend one side of a polygon beyond a vertex, the angle between the extension and the next side is an **exterior angle**. At each vertex, the interior and exterior angles lie on a straight line, so $\text{interior} + \text{exterior} = 180^\circ$.



interior + exterior = 180°

Rule: exterior angles

The exterior angles of **any polygon** always add up to 360°. Think of walking round the polygon: you turn through exterior angles and end up facing the way you started, having turned a full circle. For a **regular** polygon each exterior angle is

$$\text{exterior angle} = \frac{360^\circ}{n}$$

This gives a fast method for the number of sides: $n = \frac{360}{\text{exterior angle}}$.

Method: regular polygon problems

1. Find the exterior angle: $360 \div n$, or $180-$ interior angle.
2. Find n from $n = \frac{360}{\text{exterior angle}}$, or the interior angle from $180-$ exterior.
3. Check that n is a whole number.

Worked example 3: from the exterior angle

A regular polygon has exterior angles of 20°. How many sides?

$n = \dfrac{360}{20} = 18$ sides.

Worked example 4: from the interior angle

A regular polygon has interior angles of 150°. How many sides?

The exterior angle is $180 - 150 = 30^\circ$, so $n = \dfrac{360}{30} = 12$.

Worked example 5: an impossible polygon

Can a regular polygon have an interior angle of 100°?

The exterior angle would be 80°, and $\dfrac{360}{80} = 4.5$ is not a whole number, so **no**.

Using polygon facts in problems

Worked example 6: pentagon, tessellating shapes and algebra

(a) In a regular pentagon ABCDE find the base angles of triangle ABC. (b) A regular hexagon and a square meet at a point with a third regular polygon: how many sides does it have? (c) Five angles x, $x + 10$, $x + 20$, $x + 30$, $x + 40$ form a pentagon: find x.

(a) The interior angle is 108°, so the base angles are $\dfrac{180 - 108}{2} = 36^\circ$. (b) The remaining angle is $360 - 120 - 90 = 150^\circ$, the interior angle of a regular dodecagon (12 sides). (c) $5x + 100 = 540$, so $x = 88$.

Watch out: common mistakes

- Using $n \times 180$ instead of $(n - 2) \times 180$.
- Mixing up interior and exterior angles. Remember that they add to 180°.
- Dividing 180 by n rather than 360 for the exterior angle.
- Using the regular-polygon formulae for irregular shapes: the sum is always correct, but the angles are not equal.
- Forgetting that the number of sides must be a whole number.

Worked example 7: tiling with two shapes

Show that regular octagons and squares can be used together to tile a floor.

The interior angle of a regular octagon is $1080 \div 8 = 135^\circ$. The interior angle of a square is 90°. At a point where two octagons and one square meet, the angles add up to $135 + 135 + 90 = 360^\circ$. They fill the full turn exactly, so there is no gap and no overlap. The pattern can be repeated across the whole floor.

Worked example 8: exterior angles and algebra

A regular polygon has an interior angle that is 8 times the size of its exterior angle. How many sides does it have?

Let the exterior angle be e. Then the interior angle is $8e$ and $e + 8e = 180$, so $9e = 180$ and $e = 20^\circ$. The number of sides is $360 \div 20 = 18$.

Worked example 9: mitre joint

A picture frame is shaped like a regular hexagon. At what angle should each piece of wood be cut at its ends?

The interior angle of a regular hexagon is $720 \div 6 = 120^\circ$. Two pieces meet at each corner, so each piece is cut at half of this angle: $120 \div 2 = 60^\circ$.

Note: two ways to find the interior angle

For a regular polygon there are two routes. Route 1: find the angle sum $(n - 2) \times 180$ and divide by n. Route 2: find the exterior angle $360 \div n$ and subtract it from 180°. Route 2 is usually quicker. Use the other route to check your answer, because getting the same result both ways is strong evidence that you are right.

Watch out: it must be a whole number

- The number of sides is always a whole number. If $360\div$ (exterior angle) is not whole, the polygon does not exist.
- The exterior angles of any polygon add to 360°, but the interior angles add to $(n - 2) \times 180^\circ$, which gets bigger as n grows.
- Do not use the formula for a regular polygon on an irregular one: for an irregular polygon you can only find the sum, not each angle.

Did you know?

The Pentagon in Washington, D.C. is a regular pentagon with five sides, and honeybees build their honeycomb from regular hexagons. Mathematicians have proved that hexagons use the least wax to cover a flat area with equal cells, so bees are very efficient builders.

More practice with polygons

Worked example 10: a polygon with a reflex angle

An arrowhead (a concave quadrilateral) has angles of 30°, 30° and 40°. Find the fourth angle.

A concave polygon has at least one interior angle bigger than 180°, called a reflex

angle, but the angle sum rule still works because the shape can still be split into triangles from a suitable vertex. A quadrilateral has $n = 4$, so the angles add up to $(4 - 2) \times 180^\circ = 360^\circ$. The three known angles add up to 100°, so the fourth angle is $360^\circ - 100^\circ = 260^\circ$. This is a reflex angle, which matches the picture of an arrowhead pointing inwards. Always ask yourself whether your answer looks sensible for the shape you are given.

Worked example 11: exterior angles of an irregular polygon

Four exterior angles of a pentagon are 50°, 85°, 70° and 90°. Find the fifth exterior angle and the interior angle at the same vertex.

The exterior angles of any polygon add up to 360°, even when the polygon is not regular. The four given angles add up to 295°, so the fifth exterior angle is $360^\circ - 295^\circ = 65^\circ$. The interior and exterior angles at a vertex lie on a straight line, so the interior angle there is $180^\circ - 65^\circ = 115^\circ$.

Method: answering a multi-step polygon question

1. Write down the number of sides and say whether the polygon is regular or irregular.
2. For an irregular polygon, start with the angle sum $(n - 2) \times 180^\circ$ and subtract the angles you know.
3. For a regular polygon, it is usually quickest to find the exterior angle first with $360^\circ \div n$, then the interior angle with 180° minus the exterior angle.
4. If the question involves tiling, check whether the angles meeting at a point add up to exactly 360°.
5. Finish with a sentence that answers the question, including the units (degrees, metres or pounds).

Note: names of polygons

Many polygon names come from Greek numbers: a pentagon has five sides, a hexagon six, a heptagon seven, an octagon eight, a nonagon nine, a decagon ten and a dodecagon twelve. For other numbers mathematicians simply say "a 15-sided polygon" or "a 15-gon". Learning the names saves time in an exam, because the question often gives only the name and expects you to know the value of n.

Otto's tip

For regular polygons, find the exterior angle first. It is the easiest one. Then interior = 180° minus exterior. Hoot hoot!