Lesson - Volume and surface area of prisms

Year: Year 8

This lesson covers volume and surface area of cuboids, prisms and cylinders, working backwards and capacity.

Starter activity

Build a cuboid from 24 small cubes in as many different ways as you can (for example 2 by 3 by 4, or 1 by 4 by 6). For each one, count the number of cubes (the volume) and the number of squares on the outside (the surface area). Which shape has the smallest surface area? Which is the most spread out? What does that tell you about wrapping presents?

Volume and surface area

Definition: volume

Volume is the amount of space inside a 3-D shape and is measured in cubic units such as cm^3 or m^3.

Definition: surface area

Surface area is the total area of all the faces and is measured in square units such as cm^2. Volume is about filling; surface area is about wrapping or painting.

Cuboids and cubes

Rule: cuboid and cube

For a cuboid with length l, width w and height h:

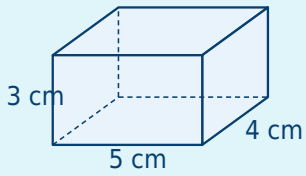
$$V = lwh \qquad SA = 2(lw + lh + wh)$$

A cube of side a has $V = a^3$ and $SA = 6a^2$.

Worked example 1: cuboid

Find the volume and surface area of a cuboid 5 cm by 4 cm by 3 cm.

$V = 5 \times 4 \times 3 = 60$ cm^3. Surface area $= 2(20 + 15 + 12) = 2 \times 47 = 94$ cm^2, because there are three pairs of identical faces. For a cube with $a = 6$, $V = 216$ cm^3 and $SA = 6 \times 36 = 216$ cm^2.



Prisms

Definition: prism

A **prism** is a 3-D shape with the same cross-section all the way through its length. Triangular prisms, trapezium prisms, L-shaped prisms and cylinders are all prisms.

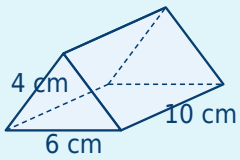
Rule: volume of a prism

$$V = \text{area of cross-section} \times \text{length}$$

Worked example 2: triangular prism

A triangular prism has a triangular end with base 6 cm and height 4 cm, and is 10 cm long. Find its volume.

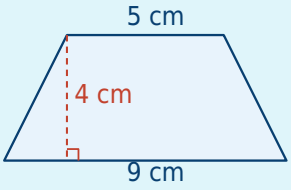
The triangle's area is $\frac{1}{2} \times 6 \times 4 = 12$ cm^2. Volume $= 12 \times 10 = 120$ cm^3.



Worked example 3: trapezium prism

The cross-section is a trapezium with parallel sides 5 cm and 9 cm and height 4 cm, and the length is 6 cm. Find the volume.

Area $= \frac{1}{2}(5+9) \times 4 = 28$ cm^2. $V = 28 \times 6 = 168$ cm^3.



Worked example 4: L-shaped prism

An L-shaped cross-section is split into a 6 cm by 2 cm rectangle and a 3 cm by 4 cm rectangle. The length is 8 cm. Find the volume.

Area $= 12 + 12 = 24$ cm^2. $V = 24 \times 8 = 192$ cm^3.

Surface area of a prism

Method: surface area of a prism

1. Sketch the **net**, which is the shape folded flat.
2. Find the area of the two equal ends.
3. Find the areas of the rectangles round the outside (together a rectangle whose width is the perimeter of the cross-section).
4. Add all the areas.

Worked example 5: surface area of a triangular prism

A prism has a right-angled triangle end with sides 3 cm, 4 cm and 5 cm and length 10 cm. Find the surface area.

The two ends have area $2 \times \frac{1}{2} \times 3 \times 4 = 12$ cm^2. The three rectangles have areas $10 \times 3 = 30$, $10 \times 4 = 40$ and $10 \times 5 = 50$, total 120 cm^2. Surface area $= 12 + 120 = 132$ cm^2.

Cylinders

Rule: cylinder

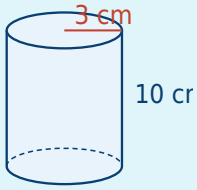
A cylinder is a prism with a circle as its cross-section. With radius r and height h:

$$V = \pi r^2 h \qquad \text{curved surface area} = 2\pi r h \qquad SA = 2\pi r h + 2\pi r^2$$

Worked example 6: cylinder

Find the volume and total surface area of a cylinder with $r = 3$ cm and $h = 10$ cm.

$V = \pi \times 9 \times 10 = 90\pi = 282.7$ cm^3. The curved surface unrolls into a rectangle of width $2\pi r$ and height h, so its area is $2\pi \times 3 \times 10 = 60\pi = 188.5$ cm^2. Add the two circles $2 \times 9\pi = 18\pi$ to get $SA = 78\pi = 245.0$ cm^2.



Working backwards and capacity

Worked example 7: finding a missing length

A cylinder has $V = 200$ cm^3 and $r = 4$ cm. Find h. A cuboid has volume 240 cm^3, length 8 cm and width 5 cm; find its height.

$h = \dfrac{V}{\pi r^2} = \dfrac{200}{16\pi} \approx 4.0$ cm. For the cuboid, $h = \dfrac{240}{40} = 6$ cm. A cube of volume 125 cm^3 has side $\sqrt[3]{125} = 5$ cm.

Note: units and capacity

Capacity is the volume of liquid a container holds. $1 \text{ litre} = 1000 \text{ cm}^3$ and $1 \text{ m}^3 = 1\,000\,000 \text{ cm}^3$. A tank 50 cm by 30 cm by 20 cm has volume $30\,000$ cm^3, which is 30 litres. If it holds 15 litres of water, the depth is $\dfrac{15\,000}{50 \times 30} = 10$ cm.

Watch out: common mistakes

- Using the slant side instead of the perpendicular height of the triangle.
- Adding the length to the area of the cross-section instead of multiplying.
- Forgetting one pair of faces when finding the surface area.
- Mixing up cm^2 and cm^3.
- Using the diameter instead of the radius in the cylinder formula.

Worked example 8: volume of a cylinder in context

A cylindrical water butt has a radius of 40 cm and a height of 90 cm. How many litres does it hold when full?

$V = \pi r^2 h = \pi \times 40^2 \times 90 = 144\,000\pi = 452\,389$ cm³. Divide by 1 000 to convert to litres: $452\,389 \div 1000 = 452$ litres (to the nearest litre).

Worked example 9: surface area of a trapezium prism

A prism has a trapezium cross-section with parallel sides 6 cm and 10 cm, height 4 cm and sloping sides 5 cm each. The prism is 12 cm long. Find the

total surface area.

Two ends: $2 \times \frac{1}{2}(6+10) \times 4 = 64$ cm². The rectangles: the perimeter of the trapezium is $6+10+5+5 = 26$ cm, so the rectangles together have area $26 \times 12 = 312$ cm². Total $= 64 + 312 = 376$ cm².

Worked example 10: finding a height from a volume

A cuboid fish tank has a base of 60 cm by 30 cm and must hold 72 litres. How high must it be?

72 litres $= 72\,000$ cm³. Base area $= 60 \times 30 = 1800$ cm². Height $= 72000 \div 1800 = 40$ cm.

Worked example 11: change in volume

A cube has side 4 cm. Each side is doubled. How does the volume change?

Original volume $= 4^3 = 64$ cm³. New side $= 8$, new volume $= 512$ cm³. The volume is multiplied by $512 \div 64 = 8$, which is 2^3. The surface area is multiplied by $2^2 = 4$.

Method: surface area of any prism

1. Draw the cross-section and find its area.
2. The two ends have a total area of twice the cross-section.
3. Find the perimeter of the cross-section and multiply by the length to get all the rectangular faces together.
4. Add the two parts and write the units in cm² or m².

Note: units

Length is in cm or m, area is in cm² or m², and volume is in cm³ or m³. When you convert units, remember that 1 m = 100 cm, 1 m² = 10 000 cm² and 1 m³ = 1 000 000 cm³. Capacity is the volume of liquid a container can hold: 1 litre = 1 000 cm³ = 1 000 ml and 1 m³ = 1 000 litres.

Note: checking your answers

Always ask whether an answer is sensible. A fish tank that holds 5 000 litres would be as big as a car, and a water bottle that holds 15 litres is too heavy to carry. Estimate by rounding each length to one significant figure, multiply, and compare with your calculator answer. Check that you have used the same unit for every length before multiplying, and that your final unit is cubic for volume and square for area.

Watch out: common mistakes

- Using the slant height instead of the perpendicular height of a triangle or trapezium.
- Forgetting the $\frac{1}{2}$ when the cross-section is a triangle.
- Mixing units, such as multiplying 2 m by 50 cm without converting.
- Giving surface area in cubic units, or volume in square units.

Did you know?

Archimedes, who lived in ancient Greece, discovered how to measure the volume of an awkward shape by lowering it into water and measuring how much water it pushed out. According to the story he shouted "Eureka!", meaning "I have found it", when he made the discovery in his bath. He asked for a sphere inside a cylinder to be carved on his tomb, because he was so proud of his work on volumes.

One idea for every prism

Cuboids, triangular prisms, L-shaped blocks and cylinders all look different, but their volumes are found in exactly the same way: area of the cross-section multiplied by the length. A cylinder is simply a prism whose cross-section is a circle, so its volume $\pi r^2 h$ is the circle's area πr^2 multiplied by the height. For surface area, imagine unfolding the solid into a flat net: two identical ends, plus a set of rectangles whose widths go all the way round the cross-section. That is why the total area of the rectangles equals the perimeter of the cross-section multiplied by the length.

Example: using the perimeter shortcut

A prism has a cross-section with area 30 cm² and perimeter 26 cm, and a length of 10 cm. Find its volume and surface area.

Volume = area of cross-section × length $= 30 \times 10 = 300$ cm³. The two ends together have area $2 \times 30 = 60$ cm². The rectangular faces wrap around the cross-section, so their total area is perimeter × length $= 26 \times 10 = 260$ cm². The total surface area is $60 + 260 = 320$ cm². This shortcut saves time whenever the cross-section has many sides.

Otto's tip

Find the area of the cross-section first, then multiply by the length. For surface area, draw the net and tick off every face so none are missed.