Lesson - Transformations

Year: Year 8

This lesson covers reflection, translation, rotation and enlargement, their effect on lengths and areas, and how to describe and combine them.

Starter activity

Look at a letter such as F or R drawn on a piece of tracing paper. Slide it across the page. Flip it over. Turn it a quarter of a turn. In each case, say which of the letter's properties changed (its position, its direction or its size) and which stayed the same. Now try to make the letter bigger without changing its shape. What do you need to know?

What is a transformation?

Definition: transformation, object and image

A **transformation** changes the position, orientation or size of a shape. The original shape is the **object** and the new shape is the **image**. We label image points with a dash, so A becomes A'.

Key fact: congruent or similar

Reflections, rotations and translations keep the shape the same size, so the image is **congruent** to the object. An enlargement changes the size, so the image is **similar** to the object, not congruent.

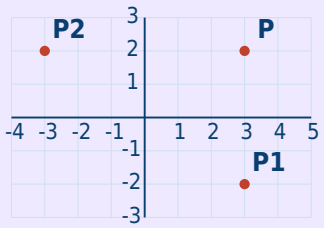
Reflection

Definition: reflection

A **reflection** flips a shape in a mirror line. Each point of the image is the same distance from the mirror line as the object point, but on the opposite side, and the line joining them is at right angles to the mirror. To *describe* a reflection, give the word "reflection" and the equation of the mirror line.

Rule: reflecting a point

- Reflect $(3, 2)$ in the x-axis: the y-coordinate changes sign, giving $(3, -2)$.
- Reflect in the y-axis: the x-coordinate changes sign, giving $(-3, 2)$.
- Reflect in the line $y = x$: swap the coordinates, giving $(2, 3)$.
- Reflect in the vertical line $x = 1$: the point $(3, 2)$ is 2 units to the right of the line, so the image is 2 units to the left, at $(-1, 2)$.



Translation

Definition: translation

A **translation** slides every point the same distance in the same direction. It is described by a column vector $\begin{pmatrix} a \\ b \end{pmatrix}$: a units right (left if negative) and b units up (down if negative). It does not turn or flip the shape.

Worked example 1: translating and finding a vector

Translate $(1, 4)$ by $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$. Find the vector from $A(1, 1)$ to $B(4, 5)$.

$(1, 4)$ goes to $(1 + 2, 4 - 3) = (3, 1)$. The vector from A to B is found by subtracting: $\begin{pmatrix} 4 - 1 \\ 5 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Rotation

Definition: rotation

A **rotation** turns a shape about a fixed point, the **centre of rotation**, through an angle in a given direction (clockwise or anticlockwise). To describe a rotation you need three things: the angle, the direction and the centre.

Rule: rotating about the origin

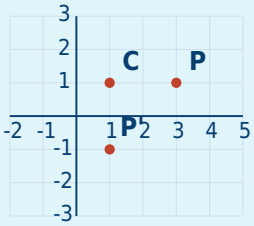
A 90° clockwise turn sends (x, y) to $(y, -x)$, a 90° anticlockwise turn sends (x, y) to $(-y, x)$, and a 180° turn sends (x, y) to $(-x, -y)$. So $(2, 1)$ becomes $(1, -2)$ after 90° clockwise, $(-1, 2)$ after 90° anticlockwise, and $(-2, -1)$ after 180°.

Method: rotating about another centre

1. Find the position of the point relative to the centre.
2. Rotate that relative position using the rule for the origin.
3. Add the centre back.

Worked example 2: rotation about another centre

Rotate $(3, 1)$ through $90°$ clockwise about $(1, 1)$.



Relative to the centre: $(3-1, 1-1) = (2, 0)$. Rotate by $90°$ clockwise using $(x, y) \to (y, -x)$: $(0, -2)$. Add the centre back: $(0+1, -2+1) = (1, -1)$.

Enlargement

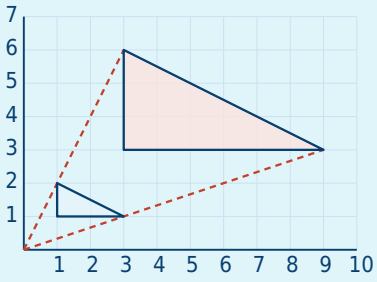
Definition: enlargement

An **enlargement** changes the size of a shape using a **scale factor** and a **centre of enlargement**. Every length is multiplied by the scale factor, but the angles stay the same. A scale factor between 0 and 1 makes the shape smaller.

Worked example 3: enlargement from the origin

Enlarge the triangle with vertices $(1, 1)$, $(3, 1)$ and $(1, 2)$ by scale factor 3 from the origin.

With centre at the origin, multiply each coordinate by the scale factor. The image has vertices $(3, 3)$, $(9, 3)$ and $(3, 6)$. A scale factor of $\frac{1}{2}$ would send $(4, 6)$ to $(2, 3)$.



Worked example 4: enlargement from another centre

Enlarge $(3, 2)$ by scale factor 2 with centre $(1, 1)$.

The vector from the centre to the point is $(2, 1)$. Multiply it by 2 to get $(4, 2)$ and add the centre: $(5, 3)$.

Effect on length, perimeter and area

Rule: scale factor k

Lengths and perimeters are multiplied by k, but areas are multiplied by k^2.

Worked example 5: scale factor and area

A length of 4 cm is enlarged to 10 cm. Find the scale factor and the new area of a region of area 12 cm^2.

The scale factor is $\frac{10}{4} = 2.5$. The area becomes $12 \times 2.5^2 = 75$ cm^2. (With scale factor 3, an area of 1 square unit becomes 9.)

Describing and combining transformations

Worked example 6: combining and describing

Reflect $(4, 1)$ in the x-axis, then translate by $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Describe the move $(1, 1) \to (2, 2)$ with $(3, 1) \to (6, 2)$.

Reflecting gives $(4, -1)$; translating gives $(3, 1)$. The second move is an enlargement with scale factor 2 centred on the origin. The move $(2, 3) \to (-2, 3)$ would be a reflection in the y-axis.

Watch out: common mistakes

- Describing a transformation with only its name and missing the details such as the mirror line, centre or angle.
- Mixing up clockwise and anticlockwise turns.
- Reflecting a point in the wrong direction when the mirror line is not an axis.
- Adding the scale factor instead of multiplying by it.
- Using k instead of k^2 for the change in area.

Worked example 7: reflect in a diagonal line

Reflect the point $(2, 5)$ in the line $y = x$.

Reflection in $y = x$ swaps the coordinates, so the image is $(5, 2)$. Check: both points are the same distance from the line $y = x$, one above and one below it, and the line joining them is perpendicular to it.

Worked example 8: rotation about another centre

Rotate the point $P(4, 3)$ by $90°$ clockwise about the centre $C(2, 1)$.

Subtract the centre: $P - C = (2, 2)$. Rotate $90°$ clockwise: $(x, y) \to (y, -x)$, so $(2, 2) \to (2, -2)$. Add the centre back: $(2 + 2, 1 - 2) = (4, -1)$. The image is $(4, -1)$.

Worked example 9: finding the centre of an enlargement

The point $(2, 3)$ is mapped to $(4, 5)$ and the point $(3, 3)$ is mapped to $(6, 5)$ by an enlargement with scale factor 2. Find the centre.

Length from $(2, 3)$ to $(3, 3)$ is 1 and from $(4, 5)$ to $(6, 5)$ is 2, so the scale factor is 2. Lines through corresponding points meet at the centre. For the image point $(4, 5) = C + 2 \times ((2, 3) - C)$, so $C = (2 \times 2 - 4, 2 \times 3 - 5) = (0, 1)$. The centre is $(0, 1)$.

Worked example 10: scale factor and volume

A cube of side 3 cm is enlarged by scale factor 2. By what factor does the volume increase?

Volume of the original $= 3^3 = 27$ cm³. New side $= 6$ cm, new volume $= 216$ cm³. The ratio is $216 \div 27 = 8 = 2^3$. Length scale factor k, area scale factor k^2, volume scale factor k^3.

Note: congruent or similar?

Reflections, rotations and translations produce an image that is *congruent* to the object: the same size and the same shape. An enlargement produces an image that is *similar*: the same shape but a different size. The image of a reflection is a mirror image, so the order of the vertices (clockwise or anticlockwise) is reversed.

Watch out: describing fully

- A translation needs a vector, a reflection needs a mirror line, a rotation needs an angle, a direction and a centre, and an enlargement needs a scale factor and a centre.
- Describe only one transformation. Do not say "reflect and translate" for a single step.
- The centre of an enlargement is not always the origin.
- Lengths are multiplied by k, but areas are multiplied by k^2.

Did you know?

Patterns made by reflections, rotations and translations decorate buildings all over the world. The Alhambra palace in Granada, Spain, has tilings that use all seventeen possible types of wallpaper pattern. The Dutch artist M. C. Escher studied these patterns and used them to draw fish, birds and lizards that fit together without gaps.

What stays the same?

A useful way to understand transformations is to ask what each one keeps unchanged. Reflections, rotations and translations keep every length and every angle, so the image is always congruent to the object; they only change its position or the way it faces. A reflection turns the shape over, so the order of its vertices goes the other way round. An enlargement keeps the angles and the shape but multiplies every length by the scale factor, so the image is similar to the object and its area is multiplied by the square of the scale factor. Knowing these facts helps you to recognise a transformation quickly and to check your drawings.

Example: identifying from the facts

A triangle with sides 3 cm, 4 cm and 5 cm is mapped to a triangle with sides 6 cm, 8 cm and 10 cm. Which type of transformation could this be?

The lengths have all doubled, so the image is not congruent: it cannot be a reflection, rotation or translation. It must be an enlargement with scale factor 2. The area of the original right-angled triangle is $\frac{1}{2} \times 3 \times 4 = 6$ cm², and the image has area $6 \times 2^2 = 24$ cm², which agrees with $\frac{1}{2} \times 6 \times 8$.

Otto's tip

Use tracing paper. Trace the shape, put your pencil point on the centre of rotation and turn the paper. For enlargements, draw rays from the centre through each vertex and measure along them.