Lesson - Probability: outcomes and sample spaces

Year: Year 8

This lesson covers the probability scale, theoretical and experimental probability, listing outcomes in sample spaces and Venn diagrams.

Starter activity

Roll a die 30 times and record the results in a tally chart. Work out the relative frequency of each number. Compare with the theoretical probability of $\frac{1}{6} \approx 0.17$. Now combine your results with the results of everyone in your class. Do the relative frequencies get closer to 0.17?

The probability scale

Definition: probability

Probability measures how likely an event is. It is a number from 0 to 1. An impossible event has probability 0, a certain event has probability 1 and an even chance has probability $\frac{1}{2}$. Probabilities can be written as fractions, decimals or percentages, never as a negative number or a number greater than 1.

Theoretical probability

Rule: equally likely outcomes

When all outcomes are **equally likely**, as in a fair coin, a fair die or a well-shuffled pack:

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

Worked example 1: a die and a bag

Find $P(5)$ and $P(\text{even})$ for a fair die. A bag has 3 red, 5 blue and 2 green counters; find $P(\text{blue})$ and $P(\text{red or green})$.

$P(5) = \frac{1}{6}$ and $P(\text{even}) = \frac{3}{6} = \frac{1}{2}$. The bag holds 10 counters, so $P(\text{blue}) = \frac{5}{10} = \frac{1}{2}$ and $P(\text{red or green}) = \frac{5}{10} = \frac{1}{2}$.

Key fact: an event not happening

The probabilities of all the possible outcomes add up to 1. So $P(\text{not } A) = 1 - P(A)$.

Worked example 2: the missing outcome

$P(A) = 0.35$ and $P(B) = 0.4$ are two of three possible outcomes. Find the third probability, and $P(\text{not rolling a } 6)$.

The third is $1 - 0.35 - 0.4 = 0.25$. The probability of not rolling a 6 is $1 - \frac{1}{6} = \frac{5}{6}$.

Experimental probability

Definition: relative frequency

When outcomes may not be equally likely, you can estimate probability by doing an experiment. The **relative frequency** is $\frac{\text{number of times the event happens}}{\text{total number of trials}}$. More trials give a more reliable estimate.

Worked example 3: is the die fair?

A die is rolled 120 times and shows a 6 on 28 occasions. Is it fair?

Relative frequency $= \frac{28}{120} \approx 0.233$. A fair die would give $120 \times \frac{1}{6} = 20$ sixes, so the die may be biased, but 120 rolls is not many, so you would want more trials.

Rule: expected number

Expected number = probability $\times$ number of trials. This is an average over many trials, not a promise for any one set of trials.

Worked example 4: expected wins

A spinner wins with probability 0.2. How many wins do you expect in 150 spins? A coin gives 63 heads in 100 tosses; how many heads in 500 tosses?

$0.2 \times 150 = 30$ wins. The estimate for heads is 0.63, so in 500 tosses expect about $0.63 \times 500 = 315$ heads.

Listing outcomes systematically

Definition: sample space

A **sample space** is the list or table of all possible outcomes. Listing them in an

organised way means none are missed.

Worked example 5: coins

List the outcomes of two coins and find $P(\text{exactly one head})$. Then find $P(\text{at least one head})$ with three coins.

Two coins give HH, HT, TH, TT, four outcomes, so $P(\text{exactly one head}) = \frac{2}{4} = \frac{1}{2}$.

Three coins give 8 outcomes, so $P(\text{at least one head}) = 1 - P(\text{TTT}) = 1 - \frac{1}{8} = \frac{7}{8}$.

Worked example 6: two dice

Two fair dice are rolled and the scores are added. Find $P(\text{sum} = 7)$ and $P(\text{sum} \geq 10)$.

Draw a 6 by 6 grid with one die along the top and the other down the side, filling in the totals. There are 36 equally likely outcomes. A total of 7 appears 6 times, so $P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}$. A total of at least 10 comes from (4,6), (5,5), (6,4), (5,6), (6,5) and (6,6), which is also $\frac{6}{36} = \frac{1}{6}$.

Rule: counting combinations

Multiply the number of choices at each stage. With 3 starters and 4 main courses, the number of possible meals is $3 \times 4 = 12$.

Sets and Venn diagrams

Definition: set, union and intersection

A **set** is a collection of items. The **union** $A \cup B$ contains everything in A or B (or both). The **intersection** $A \cap B$ contains only items in both. In a Venn diagram, overlapping circles show the intersection.

Method: filling in a Venn diagram

1. Write the number in the overlap (both sets) first.
2. Subtract it from each set total to get "only" regions.
3. Add all regions; subtract from the total to find "neither".

Worked example 7: Venn diagram

In a class of 30, 18 play football, 12 play tennis and 5 play both. Find

$P(\text{neither})$ **and** $P(\text{football or tennis})$**.**

Start with the overlap: 5 play both. Football only is $18 - 5 = 13$ and tennis only is $12 - 5 = 7$. Together these cover $13 + 5 + 7 = 25$, so 5 play neither. $P(\text{neither}) = \frac{5}{30} = \frac{1}{6}$ and $P(\text{football or tennis}) = \frac{25}{30} = \frac{5}{6}$.

Worked example 8: sets of numbers

For the numbers 1 to 12, A is the even numbers and B the multiples of 3. Find $A \cap B$**,** $A \cup B$ **and** $P(A \cup B)$**.**

$A \cap B = \{6, 12\}$ and $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12\}$, so $P(A \cup B) = \frac{8}{12} = \frac{2}{3}$.

Watch out: common mistakes

- Writing a probability as a ratio such as 1:6 instead of a fraction.
- Giving a probability greater than 1 or less than 0.
- Counting (2,3) and (3,2) as the same outcome with two dice.
- Counting the intersection twice when working with the union.
- Thinking that a die that has not landed on 6 in a while is "due" a 6.

Worked example 9: probabilities that add up to 1

A spinner can land on red, blue or green. $P(\text{red}) = 0.3$ **and** $P(\text{blue}) = 0.45$**. Find** $P(\text{green})$ **and the expected number of greens in 80 spins.**

The probabilities add up to 1, so $P(\text{green}) = 1 - 0.3 - 0.45 = 0.25$. Expected greens $= 0.25 \times 80 = 20$.

Worked example 10: a sample space table

Two spinners, each with 1, 2, 3, are spun and the scores added. Find the probability that the total is 4.

List all $3 \times 3 = 9$ outcomes: totals 2, 3, 4 / 3, 4, 5 / 4, 5, 6. Total 4 occurs three times (1+3, 2+2, 3+1). So $P(4) = \frac{3}{9} = \frac{1}{3}$.

Worked example 11: a two-circle Venn diagram

In a class of 28, 15 like pizza (P), 12 like pasta (Q) and 5 like both. How many like neither?

Both $= 5$. Pizza only $= 15 - 5 = 10$. Pasta only $= 12 - 5 = 7$. At least one $= 10 + 5 + 7 = 22$. Neither $= 28 - 22 = 6$. $P(\text{neither}) = \frac{6}{28} = \frac{3}{14}$.

Worked example 12: choosing from relative frequency

A bag has red and blue counters. Sam picks a counter 100 times (replacing it each time) and gets red 38 times. Estimate the number of red counters if there are 50 counters in total.

Relative frequency of red $= \dfrac{38}{100} = 0.38$. Expected reds $= 0.38 \times 50 = 19$. This is an estimate: the true number might be 18 or 20.

Note: how to write probabilities

Probabilities can be fractions, decimals or percentages, but never negative or greater than 1. Write fractions in their simplest form. Use the words "impossible", "unlikely", "evens", "likely" and "certain" to describe the probability scale. To show working, write the favourable outcomes over the total number of equally likely outcomes, for example $P(\text{even}) = \dfrac{3}{6} = \dfrac{1}{2}$.

Watch out: common mistakes

- Assuming the outcomes are equally likely when they are not: for the total of two dice the totals are *not* equally likely, but each of the 36 pairs is.
- Writing probabilities as "3 in 8" instead of $\dfrac{3}{8}$.
- Forgetting to subtract the overlap when finding $n(A \cup B)$.
- Thinking that a die is "due" a six after a long run without one: each roll is independent.

Did you know?

The mathematics of probability began in 1654, when a French gambler asked the mathematician Blaise Pascal about a dice game. Pascal wrote letters to Pierre de Fermat about the problem, and between them they laid the foundation of probability theory. Today probability is used in weather forecasts, insurance, medicine and the algorithms that suggest the next video you will watch.

Choosing the right tool

In this chapter you have met several ways to organise outcomes, and choosing the right one makes a problem much easier. A simple list works well for a single event or for two coins. A sample space grid is best when two things happen together and you want to combine them, such as adding or multiplying the scores on two dice. A Venn diagram is best when people or numbers belong to overlapping groups, such as pupils who play football, tennis or both. Whatever you choose, check that every outcome appears exactly once and that the probabilities add up to 1.

Example: theory or experiment?

A drawing pin lands point up 62 times in 200 drops. A fair coin lands heads 104 times in 200 flips. Which probabilities should we use?

For the coin we can calculate a theoretical probability, $\frac{1}{2}$, because heads and tails are equally likely; the 104 heads (0.52) is just normal variation. For the drawing pin the two outcomes are not equally likely, so there is no theoretical value and the relative frequency $\frac{62}{200} = 0.31$ is our best estimate. Repeating the experiment more times would make this estimate more reliable.

Did you know?

Weather forecasters use probability every day. A “70% chance of rain” means that, on days with similar conditions in the past, it rained on about 70 out of every 100 of them.

Otto's tip

Make a list or a table before you count. Check that the probabilities of all the outcomes add up to 1. If they do not, something is missing or counted twice.