Lesson - Powers, roots and standard form

Year: Year 8

This lesson covers powers, roots, the index laws, prime factors and standard form. Learn the squares up to 15^2 and the cubes of 1 to 5 and 10 first.

Starter activity

Write down the number of seconds in a day. It is a big number: 86 400. Now write the number of seconds in a year. How many digits does your answer have? Why is it easier to write 3.15 $\times 10^7$ than 31 536 000? Try the same for a very tiny thickness, the width of a spider's thread, which is about 0.000 003 m.

Powers and roots

Definition: power, base and index

A **power** (or index) tells you how many times to multiply a number by itself. In 2^5 the **base** is 2 and the **index** is 5, so $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. Recognise powers of 2 (2, 4, 8, 16, 32, 64, 128), of 3 (3, 9, 27, 81) and of 5 (5, 25, 125).

Definition: root

A **root** is the inverse of a power. The square root $\sqrt{144} = 12$ because $12^2 = 144$. The cube root $\sqrt[3]{27} = 3$ because $3^3 = 27$.

Worked example 1: estimating a root

Estimate $\sqrt{50}$.

$7^2 = 49$ and $8^2 = 64$, so $\sqrt{50}$ lies between 7 and 8, and it is very close to 7 (about 7.07).

The index laws

Rule: the index laws

For any non-zero base a:

- **Multiplying:** $a^m \times a^n = a^{m+n}$. Example: $2^3 \times 2^4 = 2^7 = 128$.
- **Dividing:** $a^m \div a^n = a^{m-n}$. Example: $5^9 \div 5^6 = 5^3 = 125$.
- **Power of a power:** $(a^m)^n = a^{mn}$. Example: $(3^2)^3 = 3^6 = 729$.
- **Zero index:** $a^0 = 1$. Example: $7^0 = 1$.

- **Negative index:** $a^{-n} = \frac{1}{a^n}$. Example: $2^{-3} = \frac{1}{8}$.

Key fact: why the laws work

Write $2^3 \times 2^4$ out in full: $(2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2)$ has seven twos, so it equals 2^7. Write $5^9 \div 5^6$ as a fraction and cancel six fives, leaving 5^3. The zero index follows too: $a^3 \div a^3 = 1$, and the law says $a^{3-3} = a^0$, so $a^0 = 1$.

Watch out: same base only

The multiplying and dividing laws only work when the bases are the same, and the base stays the same: $2^3 \times 2^4 = 2^7$, not 4^7. A negative index does not make the answer negative: it means "reciprocal".

Worked example 2: using the laws

Simplify $\frac{2^5 \times 2^{-2}}{2^2}$.

Top line: $2^5 \times 2^{-2} = 2^{5+(-2)} = 2^3$. Divide: $2^3 \div 2^2 = 2^{3-2} = 2^1$. Answer: 2.

Worked example 3: a single power

Write $\frac{3^7 \times 3^2}{3^4}$ as a single power of 3, then find its value.

Top: $3^{7+2} = 3^9$. Divide: $3^{9-4} = 3^5$. The value is $3 \times 3 \times 3 \times 3 \times 3 = 243$.

Prime factors

Definition: prime factor

Every integer above 1 is a product of **prime factors**. Use powers to write the product neatly. A factor tree gives the same result however you start.

Worked example 4: prime factors of 72

Write 72 as a product of prime factors.

$72 = 2 \times 36 = 2 \times 2 \times 18 = 2 \times 2 \times 2 \times 9 = 2^3 \times 3^2$.

Standard form

Definition: standard form

Standard form writes very large or very small numbers as $A \times 10^n$, where $1 \leq A$ and n is an integer.

- Large numbers have a positive n: $45\,000 = 4.5 \times 10^4$.
- Small numbers have a negative n: $0.0006 = 6 \times 10^{-4}$.

Method: converting to and from standard form

1. Count how many places the decimal point moves to get a number A between 1 and 10.
2. Moving it left gives a positive power; moving it right gives a negative power.
3. To change back, move the point the number of places shown by n: $3.2 \times 10^5 = 320\,000$.

Worked example 5: multiplying and dividing

Work out $(2 \times 10^3) \times (3 \times 10^4)$ and $(8 \times 10^6) \div (2 \times 10^2)$.

Multiply the numbers: $2 \times 3 = 6$. Add the powers: $10^3 \times 10^4 = 10^7$. Answer: 6×10^7. For the division, $8 \div 2 = 4$ and $10^6 \div 10^2 = 10^4$, so the answer is 4×10^4.

Worked example 6: adjusting the answer

Work out $(4 \times 10^3) \times (5 \times 10^2)$.

$4 \times 5 = 20$, so the product is 20×10^5. But 20 is not between 1 and 10, so write $20 = 2 \times 10^1$. The answer is 2×10^6.

Worked example 7: adding

Work out $(6 \times 10^5) + (3 \times 10^4)$.

Change both numbers to the same power of 10 first: $3 \times 10^4 = 0.3 \times 10^5$. Then $6 + 0.3 = 6.3$, so the total is 6.3×10^5.

Worked example 8: light from the Sun

Light travels at about 3×10^5 km per second and the Sun is about 1.5×10^8 km away. How long does light take to arrive?

Time = distance $\div$ speed $= \dfrac{1.5 \times 10^8}{3 \times 10^5} = 0.5 \times 10^3 = 5 \times 10^2 = 500$ seconds, a little over 8 minutes.

Note: calculator tips

To enter 3.2×10^5, type 3.2, press the exponent key (EXP or $\times 10^x$), then 5. Do not type the multiplication sign and the 10 as well. A large number with a small raised number at the end in the display is standard form.

Watch out: common mistakes

- Writing 2^3 as $2 \times 3 = 6$ instead of $2 \times 2 \times 2 = 8$.
- Multiplying the bases when applying a law, such as $2^3 \times 2^4 = 4^7$.
- Leaving A outside the range 1 to 10, for example 25×10^3.
- Using a positive power for a number smaller than 1.
- Forgetting that $a^0 = 1$, not 0.

Worked example 9: roots and powers in context

A square patio has an area of 144 m². Find the side length, and then the length of fencing needed round it.

Side length $= \sqrt{144} = 12$ m. Perimeter $= 4 \times 12 = 48$ m. If the patio is a cube-shaped sandbox with volume 27 m³, the side is $\sqrt[3]{27} = 3$ m.

Worked example 10: negative indices

Work out $\dfrac{2^{-1} \times 2^5}{2^3}$.

Top: $2^{-1+5} = 2^4$. Divide: $2^{4-3} = 2^1 = 2$. Check with numbers: $\dfrac{1}{2} \times 32 = 16$ and $16 \div 8 = 2$.

Worked example 11: lowest common multiple with primes

Find the LCM of 24 and 36.

$24 = 2^3 \times 3$ and $36 = 2^2 \times 3^2$. For the LCM take the highest power of each prime: $2^3 \times 3^2 = 8 \times 9 = 72$. For the HCF take the lowest power of each common prime: $2^2 \times 3 = 12$.

Worked example 12: standard form with a calculator

Work out $(2.4 \times 10^5) \times (5 \times 10^{-8})$.

Multiply the numbers: $2.4 \times 5 = 12$. Add the powers of ten: $10^{5+(-8)} = 10^{-3}$. So the answer is 12×10^{-3}, which is not in standard form. Move the decimal point: $12 = 1.2 \times 10^1$, so the answer is 1.2×10^{-2}. Check: $2.4 \times 10^5 = 240\,000$ and $5 \times 10^{-8} = 0.00000005$, and $240\,000 \times 0.00000005 = 0.012$.

Worked example 13: adding in standard form

Work out $3.2 \times 10^4 + 5 \times 10^3$.

Write both with the same power of ten: $5 \times 10^3 = 0.5 \times 10^4$. Then $3.2 \times 10^4 + 0.5 \times 10^4 = 3.7 \times 10^4$. You cannot just add the numbers at the front when the powers are different.

Method: simplifying with the index laws

1. Check that every power has the *same base*. If not, try to rewrite them, for example $4 = 2^2$ and $8 = 2^3$.
2. Deal with the multiplying: add the indices on the top line.
3. Deal with the dividing: subtract the bottom index from the top index.
4. Write the answer as a single power, and then find its value if the question asks for a number.

Note: significant figures in standard form

When a question asks for an answer to 3 significant figures, the number in front of the power of ten must have 3 significant figures. For example, 4.0375×10^{13} becomes 4.04×10^{13}. Keep every digit in your calculator while you work, and round only at the very end, because early rounding can change the last digit of your answer. Always write down the units, such as km, kg or seconds, and check that the size of your answer is sensible.

Watch out: the common traps

- 2^3 means $2 \times 2 \times 2 = 8$, not $2 \times 3 = 6$.
- $2^3 \times 2^4 = 2^7$, but $2^3 + 2^4$ cannot be simplified in this way.
- $(-3)^2 = 9$, but $-3^2 = -9$ on a calculator.
- A number in standard form must have $1 \leq A$, so 12×10^3 is not allowed.

Did you know?

In 1938 the American mathematician Edward Kasner asked his nine-year-old nephew to give a name to a very large number: 1 followed by 100 zeros. The boy called it a *googol*, written 10^{100}. The search company Google took its name, with a spelling change, from this idea. A googol is bigger than the number of atoms in the observable universe, which is about 10^{80}. Scientists need standard form because the universe contains numbers from the incredibly tiny, such as the mass of an electron, to the astonishingly huge, such as the distance across our galaxy, and both fit neatly on one line.

Why the index laws work

The index laws are not rules to memorise blindly: each one comes from writing the powers out in full. For example, $2^3 \times 2^4$ means $(2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2)$, which is seven 2s multiplied together, so it equals 2^7. In the same way $5^6 \div 5^4$ is six 5s on top and four 5s underneath; four pairs cancel and 5^2 is left. Following the same pattern down, $5^2 \div 5^2 = 1$ shows why any number to the power 0 is 1, and $5^2 \div 5^3 = \frac{1}{5}$ shows why 5^{-1} means one fifth.

Example: standard form with a calculator

Work out $(3.6 \times 10^5) \div (4.5 \times 10^{-2})$.

Divide the numbers and the powers of 10 separately: $3.6 \div 4.5 = 0.8$ and $10^5 \div 10^{-2} = 10^{5-(-2)} = 10^7$. The result 0.8×10^7 is not in standard form, because 0.8 is less than 1, so we write it as 8×10^6. Always check that the first number is at least 1 and strictly less than 10 before you write your final answer.

Did you know?

A googol is 10^{100}, a 1 followed by a hundred zeros. The name was invented in 1920 by a nine-year-old boy, the nephew of an American mathematician, and the company Google later took its name from a misspelling of it.

Otto's tip

When the bases match, look at the powers only. Adding, subtracting or multiplying small whole numbers is much easier than working with giant values. Finish by checking that the first number in standard form is at least 1 and less than 10.