Lesson - Constructions and congruence

Year: Year 8

This lesson covers ruler and compass constructions, then congruent triangles and how to prove them. Keep your construction arcs visible, because they show the examiner how you made the shape.

Starter activity

Take a strip of card and cut three pieces of length 6 cm, 8 cm and 10 cm. Join them with paper fasteners. Can you push the joints to change the shape? Now try the same with four pieces of card joined in a rectangle. What is the difference, and why do builders use triangles in roofs and bridges?

Constructing triangles

Definition: construction

A **construction** is an accurate drawing made with only a pair of compasses, a straight edge (ruler) and a pencil. Every step is exact and the reason it works can be explained.

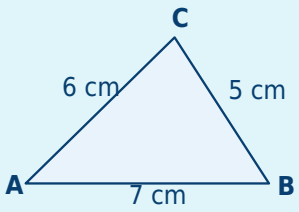
Method: constructing an SSS triangle

1. Draw the longest side as the base, for example $AB = 7$ cm.
2. Set the compasses to the length of the second side (6 cm), centre A, and draw an arc above AB.
3. Set the compasses to the third side (5 cm), centre B, and draw an arc that crosses the first one. Call the crossing point C.
4. Join AC and BC with the ruler.

Worked example 1: SSS triangle

Construct triangle ABC with $AB = 7$ cm, $BC = 5$ cm and $CA = 6$ cm.

Follow the four steps of the method above. The arcs cross at C, and joining A, B and C gives the triangle.



Key fact: when a triangle exists

Three lengths form a triangle only if the two shorter sides add up to more than the longest. So 5, 6 and 10 can make a triangle ($5 + 6 = 11 > 10$), but 3, 4 and 8 cannot ($3 + 4 = 7$), and the arcs would not cross.

The perpendicular bisector

Definition: perpendicular bisector

The **perpendicular bisector** of a line segment AB cuts it in half at right angles.

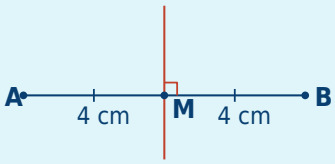
Method: perpendicular bisector

1. Open the compasses to more than half of AB. Put the point on A and draw an arc above and below the line.
2. With the same radius, put the point on B and draw arcs that cross the first two.
3. Join the two crossing points with a ruler. This line passes through the midpoint of AB and makes an angle of 90°.

Worked example 2: bisecting a line of 8 cm

Construct the perpendicular bisector of $AB = 8$ cm.

Use the three steps above. The line meets AB at its midpoint, 4 cm from each end. Every point on the bisector is the same distance from A as from B, which makes it useful when you need a point equidistant from two places.



Bisecting an angle

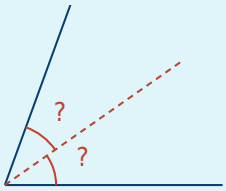
Method: bisecting an angle

1. With centre at the vertex, draw an arc cutting both arms of the angle.
2. With the compasses centred on each of these two points in turn, and the same radius, draw arcs that cross inside the angle.
3. Join the vertex to the crossing point. This line splits the angle into two equal halves.

Worked example 3: bisecting angles

Bisect an angle of 70°. What size is each half? What about a 112° angle?

A 70° angle becomes two angles of 35°. A 112° angle becomes two angles of 56°.



Perpendiculars and 60 degrees

Method: perpendicular from a point to a line

1. Draw an arc centred at P that crosses the line twice.
2. Bisect the line between those two crossing points using the perpendicular bisector method.
3. The line through P and the bisector point is at 90° to the original line.

To construct a perpendicular *at* a point on the line, mark equal distances either side of the point first and then bisect.

Note: constructing 60°

Draw a line and mark a point. Draw an arc from the point that crosses the line, then use the same radius from the crossing point to cross the arc. Joining the original point to this new point makes an angle of 60°, because the three points form an equilateral triangle.

Congruent triangles

Definition: congruent

Two shapes are **congruent** if they are exactly the same shape and size. One can be turned, reflected or moved to fit exactly on the other. Matching sides and angles are called **corresponding** parts.

Rule: the four congruence tests

- **SSS:** three sides equal.
- **SAS:** two sides and the *included* angle (the angle between them) equal.
- **ASA** (or AAS): two angles and a corresponding side equal.
- **RHS:** a right angle, the hypotenuse and one other side equal.

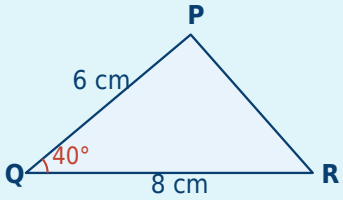
Watch out: AAA is not enough

Three equal angles (AAA) is *not* a congruence test: two equilateral triangles of different sizes both have angles of 60° but are not congruent. They are similar, not congruent.

Worked example 4: choosing a test

Triangle PQR has $PQ = 6$, $QR = 8$ and angle $Q = 40^\circ$. Triangle XYZ has $XY = 6$, $YZ = 8$ and angle $Y = 40^\circ$. Are they congruent?

The two given sides surround the given angle in both triangles, so the triangles are congruent by SAS.



Worked example 5: finding missing values

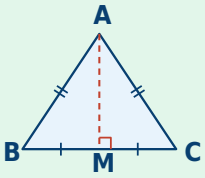
Triangle ABC is congruent to triangle PQR, with A matching P, B matching Q and C matching R. Angle $A = 40^\circ$ and angle $B = 65^\circ$. Find angle R.

Angle $C = 180 - 40 - 65 = 75^\circ$, so angle $R = 75^\circ$. The order of the letters tells you which parts match.

Using congruence in reasoning

Property: base angles of an isosceles triangle

In an isosceles triangle ABC with $AB = AC$, draw a line from A to the midpoint M of BC. Triangles ABM and ACM have $AB = AC$, $BM = CM$ and the shared side AM, so they are congruent by SSS. It follows that angle B equals angle C: the base angles of an isosceles triangle are equal.



Not drawn accurately

Worked example 6: right-angled triangles

Two right-angled triangles each have hypotenuse 13 cm and one side 5 cm.

Are they congruent?

The third side of each is $\sqrt{169 - 25} = 12$ cm. They also match by RHS (right angle, hypotenuse, one side), so they are congruent.

Watch out: common mistakes

- Changing the compass setting between the two arcs on a bisector.
- Rubbing out the construction arcs.
- Using SSA or AAA as if they proved congruence.
- Writing the letters in the wrong order, so corresponding parts do not match.
- Measuring angles with a protractor when a construction is requested.

Worked example 7: a triangle that fails

A pupil says that sides of 4 cm, 6 cm and 11 cm make a triangle. Show that the pupil is wrong.

The two shorter sides add up to $4 + 6 = 10$ cm. This is less than the longest side, 11 cm. When you draw arcs of radius 4 cm and 6 cm from the two ends of the 11 cm line, they cannot meet. So no triangle exists.

Worked example 8: angle bisector problem

Two paths meet at a corner with an angle of 64°. A bench is to be placed the same distance from both paths. Describe where it goes and find the angle it makes with each path.

The set of points the same distance from two lines is the angle bisector. The bisector splits 64° into two angles of $64 \div 2 = 32^\circ$. So the bench is on the line from the corner at 32° to each path.

Worked example 9: proving congruence

In the diagram $AB = DC$ and $AD = BC$. Prove that triangles ABC and CDA are congruent.

$AB = DC$ (given). $BC = DA$ (given). AC is a common side to both triangles. The triangles have three pairs of equal sides, so they are congruent by SSS. Always write the reason beside each fact: given, common side, vertically opposite angles.

Note: writing a proof

A good congruence proof has three parts. First list each pair of equal parts with a reason. Second, state the test (SSS, SAS, ASA or RHS) in capital letters. Third, say what the congruence tells you, such as *so angle* B = *angle* D. Write the vertices in matching order, so that ABC and PQR mean that A corresponds to P.

Watch out: congruent or similar?

- Congruent triangles are identical in shape and size. Similar triangles have the same shape but may be different sizes.
- Equal angles alone (AAA) never prove congruence.
- For SAS the angle must be *between* the two sides. Two sides and a non-included angle (SSA) can give two different triangles.

Did you know?

Surveyors have used congruent triangles for thousands of years. The Egyptians re-measured field boundaries after each flood of the River Nile, and today satellites use triangulation to find your position: your phone measures its distance from at least three satellites, much as you used arcs from three points to find where they cross.

Note

When you write a congruence proof, list the three pairs of equal sides or angles, give a reason for each one, and finish by naming the condition (SSS, SAS, ASA or RHS) that proves the triangles congruent. Keep the letters of the vertices in matching order so that corresponding points are easy to see.

Otto's tip

Check a construction by measuring. The perpendicular bisector should meet the line at 90 degrees, and each half should be the same length.