Lesson - Vectors in three dimensions

Year: Year 13

In two dimensions a vector has two components. In **three dimensions** we add a third axis, the z-axis, and a third unit vector $\mathbf{k}$.

Starter activity

Stand in a corner of your classroom and imagine a straight line to the opposite corner of the ceiling. How could you calculate its length from the dimensions of the room? Draw a sketch showing a right-angled triangle on the floor and a second right-angled triangle that stands upright. Explain how Pythagoras is used twice.

Vectors in three dimensions

Definition: 3D vector and position vector

Every vector can be written $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or as a column $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ have length 1 and point along the three perpendicular axes. The **position vector** of $P(x, y, z)$ is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Rule: adding and scaling

Add and subtract component by component; a scalar multiplies every component.

Worked example 1

$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$ and $2\mathbf{a} - \mathbf{b}$.

$\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $2\mathbf{a} - \mathbf{b} = 3\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}$.

Magnitude and unit vectors

Rule: magnitude and unit vector

By applying Pythagoras twice, $|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$. A **unit vector** in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$. For a vector of length L in that direction, multiply the unit vector by L.

Worked example 2

Find $|3\mathbf{i}-4\mathbf{j}+12\mathbf{k}|$, the unit vector, and a vector of magnitude 26 in this direction.

$\sqrt{9+16+144}=13$. The unit vector is $\frac{1}{13}(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})$, and the vector of magnitude 26 is $6\mathbf{i}-8\mathbf{j}+24\mathbf{k}$.

Worked example 3

If $|2\mathbf{i}+\lambda\mathbf{j}+3\mathbf{k}|=7$, find λ.

$4+\lambda^2+9=49$, so $\lambda^2=36$ and $\lambda=\pm 6$.

Vectors between points

Rule: vector, distance and midpoint

: always "end minus start". The distance is , the midpoint has position vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$, and the point dividing AB in ratio $m:n$ is $\mathbf{a}+\frac{m}{m+n}(\mathbf{b}-\mathbf{a})$.

Worked example 4

$A(1,2,3)$, $B(4,-2,15)$. Find AB, the midpoint and the point P with $AP:PB=1:2$.

. Midpoint $\left(\frac{5}{2},0,9\right)$. $\mathbf{p}=\mathbf{a}+\frac{1}{3}(\mathbf{b}-\mathbf{a})=(2,\frac{2}{3},7)$.

Parallel vectors and collinear points

Definition: parallel and collinear

Vectors are **parallel** if $\mathbf{u}=k\mathbf{v}$. Points A, B, C are **collinear** if and are parallel: they share B, so all three lie on one line.

Worked example 5

Show that $A(1,2,3)$, $B(3,5,7)$, $C(7,11,15)$ are collinear.

and . They are parallel and share B, so the points are collinear with $AB:BC=1:2$.

Worked example 6

Find p and q so that $p\mathbf{i}+6\mathbf{j}+9\mathbf{k}$ is parallel to $2\mathbf{i}+q\mathbf{j}+3\mathbf{k}$.

Equate ratios: $\frac{p}{2}=\frac{6}{q}=\frac{9}{3}=3$, so $p=6$ and $q=2$.

Geometric problems

Method: geometry with 3D vectors

1. Parallelogram $ABCD$: use , so $\mathbf{d}=\mathbf{a}+\mathbf{c}-\mathbf{b}$.
2. Classify a triangle by calculating the three side lengths, and test a right angle with Pythagoras.
3. Use the cosine rule for an angle: $\cos B=\dfrac{AB^2+BC^2-AC^2}{2\,AB\cdot BC}$.

Worked example 7

$A(1,0,2)$, $B(3,1,4)$, $C(3,2,1)$. Classify the triangle and find $\angle ABC$.

$AB=3$, $AC=3$, $BC=\sqrt{10}$: isosceles. $\cos B=\dfrac{9+10-9}{2\times 3\times\sqrt{10}}=0.527$, so $\angle ABC=58.2^\circ$.

Worked example 8

Show that $O(0,0,0)$, $B(2,2,1)$, $C(-2,1,2)$ form a right angle at O. Find the space diagonal of a 3 by 4 by 12 cuboid.

$OB^2+OC^2=9+9=18=BC^2$, so $\angle O=90^\circ$. The space diagonal is $\sqrt{9+16+144}=13$, at $\tan^{-1}\frac{12}{5}=67.4^\circ$ to the base.

Vectors in mechanics

Worked example 9

Forces $\mathbf{F}_1=(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$ N and $\mathbf{F}_2=(\mathbf{i}-\mathbf{j}+4\mathbf{k})$ N act on a particle in equilibrium with $\mathbf{F}_3$. Find $\mathbf{F}_3$ and its magnitude.

$\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3=\mathbf{0}$ gives $\mathbf{F}_3=(-3\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ N, of magnitude $\sqrt{22}=4.69$ N.

Worked example 10

A particle has position $\mathbf{r}=(1+2t)\mathbf{i}+(2-t)\mathbf{j}+(3+2t)\mathbf{k}$. Find its velocity and

speed.

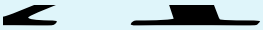
The velocity is the constant $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and the speed is 3 m/s. Two particles collide if they are at the same position at the same time: equate all three components for the same t.

Watch out


, not $\mathbf{a} - \mathbf{b}$. Do not forget the z-component in a magnitude. Give the vector between points, not a position vector, when asked. Check all three components for collinearity or collision.

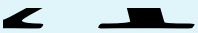
Worked example 11: Direction vectors of a given length

A drone is at $A(1, 2, 3)$ and flies towards $B(7, 5, 5)$ at a speed of 14 m/s. Write its velocity as a vector.


with magnitude $\sqrt{36 + 9 + 4} = 7$. The unit vector is $\frac{1}{7}(6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$. Multiply by the speed: $\mathbf{v} = 14 \times \frac{1}{7}(6, 3, 2) = 12\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$ m/s. Check: $\sqrt{144 + 36 + 16} = 14$.

Worked example 12: Finding a point on a line

Find the coordinates of the point on the line from $A(2, 0, 1)$ to $B(8, 6, 4)$ where the third coordinate is 3.


. A general point is 
. The third coordinate is 3, so $1 + 3\lambda = 3$ and $\lambda = \frac{2}{3}$. The point is $(2 + 4, 4, 3) = (6, 4, 3)$. Since 0 the point lies between A and B.

Worked example 13: Proving a right angle with Pythagoras

Show that triangle PQR with $P(1, 1, 1)$, $Q(3, 2, 3)$ and $R(2, 3, 1)$ is right-angled.

$PQ^2 = 4 + 1 + 4 = 9$, $PR^2 = 1 + 4 + 0 = 5$ and $QR^2 = 1 + 1 + 4 = 6$. Check which side is largest: $PQ^2 = 9$. Is $PR^2 + QR^2 = 5 + 6 = 11$? No. The sides are not a Pythagorean triple, so the triangle is **not** right-angled. Always test the largest side squared against the sum of the other two squares.

Worked example 14: A moving particle reaching a plane

A particle moves with $\mathbf{r} = (2 + t)\mathbf{i} + (1 - t)\mathbf{j} + (12 - 3t)\mathbf{k}$. Find when and where it reaches the ground, $z = 0$.

$12 - 3t = 0$ gives $t = 4$. Substituting, $\mathbf{r} = 6\mathbf{i} - 3\mathbf{j} + 0\mathbf{k}$, so the particle lands at $(6, -3, 0)$. Its speed is $\sqrt{1 + 1 + 9} = \sqrt{11} = 3.32$ m/s and the distance travelled is $4\sqrt{11} = 13.3$ m.

Worked example 15: Collision of two particles with a parameter

Particles P and Q have position vectors $\mathbf{r}_P = (2 + t)\mathbf{i} + (3 - t)\mathbf{j} + 5\mathbf{k}$ and $\mathbf{r}_Q = (8 - 2t)\mathbf{i} + (p + t)\mathbf{j} + (5)\mathbf{k}$. They collide. Find p and the time of collision.

The $\mathbf{i}$ components give $2 + t = 8 - 2t$, so $t = 2$. The $\mathbf{j}$ components must also be equal at $t = 2$: $3 - 2 = p + 2$, so $p = -1$. The $\mathbf{k}$ components are both 5. The collision occurs at $t = 2$ at the point $(4, 1, 5)$.

Note: the axes and the right-hand rule

The three axes x, y, z are perpendicular and form a *right-handed* set: if you curl the fingers of your right hand from the x-axis towards the y-axis, your thumb points along the z-axis. In most questions $\mathbf{k}$ points vertically upwards, so the z-coordinate is the height above the origin. Always read the question carefully to see which direction is "up". Distances in a vertical plane combine with horizontal distances using Pythagoras, so the space diagonal of a cuboid with edges a, b, c has length $\sqrt{a^2 + b^2 + c^2}$.

Worked example 16: Finding an unknown coordinate from a distance

The point P has coordinates $(3, -1, p)$ and the origin is O. Given that $OP = \sqrt{35}$, find the possible values of p.

[redacted], so $p^2 = 25$ and $p = \pm 5$. There are two possible points, $(3, -1, 5)$ and $(3, -1, -5)$, one above and one below the xy-plane. When a question says "find the possible values", remember the negative root.

Worked example 17: Using a ratio to find an endpoint

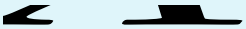
$M(3, 4, 5)$ is the midpoint of AB and A is $(1, 0, 2)$. Find B.

The midpoint formula gives $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, so $\mathbf{b} = 2\mathbf{m} - \mathbf{a} = (6, 8, 10) - (1, 0, 2) = (5, 8, 8)$. Check: the midpoint of $(1, 0, 2)$ and $(5, 8, 8)$ is $(3, 4, 5)$. Equivalently, [redacted] and [redacted] must be the same vector, so $B = M + (2, 4, 3) = (5, 8, 8)$.

Worked example 18: Finding the angle between a line and a plane

A ski lift cable runs from $A(0, 0, 10)$ to $B(60, 80, 60)$ (metres, $\mathbf{k}$ upwards). Find

the length of the cable and its angle of elevation.



. Length $= \sqrt{3600 + 6400 + 2500} = \sqrt{12500} = 111.8$ m. The horizontal distance is $\sqrt{60^2 + 80^2} = 100$ m and the rise is 50 m, so $\tan\theta = \dfrac{50}{100} = 0.5$ and $\theta = 26.6°$. The cable rises by 1 m for every 2 m horizontally.

Worked example 19: Showing that a triangle is isosceles

Show that triangle ABC with $A(1, 1, 1)$, $B(3, 3, 2)$ and $C(3, -1, 2)$ is isosceles.

$AB = \sqrt{4 + 4 + 1} = 3$. $AC = \sqrt{4 + 4 + 1} = 3$. $BC = \sqrt{0 + 16 + 0} = 4$. Two sides are equal, so the triangle is isosceles with apex A. Angle at A: $\cos A = \dfrac{9 + 9 - 16}{2 \times 3 \times 3} = \dfrac{1}{9}$, so $A = 83.6°$.

Watch out: common errors with vectors



- Subtracting in the wrong order: , not $\mathbf{a} - \mathbf{b}$. A sketch showing the direction of the arrow prevents this.
- Forgetting the third component when finding a magnitude, or forgetting the square root.
- Claiming that points are collinear because two vectors are parallel without noting that they share a common point.
- Equating position vectors at different times: for a collision the same value of t must satisfy every component.
- Leaving a vector answer as a pair of numbers when a point or a scalar is requested. Read whether the question asks for a position vector, a vector, a coordinate pair or a distance.

Worked example 20: Showing that two lines of motion cross but particles do not collide

Particle A has position $\mathbf{r}_A = (t)\mathbf{i} + (2t)\mathbf{j} + (3)\mathbf{k}$ and particle B has position $\mathbf{r}_B = (4 - t)\mathbf{i} + (2)\mathbf{j} + (3)\mathbf{k}$. Do they collide?

Equate the $\mathbf{i}$ components: $t = 4 - t$, so $t = 2$. Equate the $\mathbf{j}$ components: $2t = 2$, so $t = 1$. The two values differ, so the particles are never at the same point at the same time. However, the paths do cross at $(1, 2, 3)$: particle A is there at $t = 1$, and B is there at $t = 3$. The paths meet but the particles arrive at different times, so there is no collision.

Method: solving geometry problems in three dimensions

1. Write the position vector of every point and label the vectors you need as "end minus start".
2. Use the magnitude formula $\sqrt{x^2 + y^2 + z^2}$ for lengths, and the cosine rule for angles in triangles.
3. For parallelism or collinearity, write one vector as a scalar multiple of another.
4. For motion, equate all three components at the same time for a collision; minimise d^2, not d, to find a closest approach.
5. Finish by answering in the context, with units.

Note: why minimise the square of the distance

The distance $d = \sqrt{(\ldots)}$ is awkward to differentiate, but d^2 is a quadratic in t and d is smallest exactly when d^2 is smallest, because the square root is an increasing function. So find t from $\dfrac{d(d^2)}{dt} = 0$ and then take the square root at the end.

Did you know?

The three unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ were introduced in the 1840s by the Irish mathematician William Rowan Hamilton, who carved the key formula for his quaternions into Broom Bridge in Dublin as he walked past. Today three-dimensional vectors are used by computer-game designers to move characters and cameras, and by air-traffic controllers to keep aircraft apart.

Otto's tip

Draw a quick sketch and label the points before you calculate. It stops sign errors with end-minus-start.