Worksheet - Vectors in three dimensions

Year: Year 13 · 21

Show your working.

Question 1

Question 1 – Adding and scaling vectors *(Section A · Skills practice)*

$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

a. Find $\mathbf{a} + \mathbf{b}$. [1 mark]
b. Find $3\mathbf{a} - 2\mathbf{b}$. [2 marks]
c. Find $|\mathbf{a}|$ and $|\mathbf{a} + \mathbf{b}|$, giving answers to 3 significant figures. [3 marks]

Question 2

Question 2 – Magnitude and unit vectors *(Section A · Skills practice)*

Let $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.

a. Find $|\mathbf{v}|$. [1 mark]
b. Write down the unit vector in the direction of $\mathbf{v}$. [1 mark]
c. Find a vector of magnitude 21 in the direction of $\mathbf{v}$. [2 marks]
d. Given that $|4\mathbf{i} - \lambda\mathbf{j} + 8\mathbf{k}| = 12$, find the possible values of λ. [3 marks]

Question 3

Question 3 – The vector between two points *(Section A · Skills practice)*

A is the point $(2, -1, 4)$ and B is the point $(5, 3, -8)$.

a. Find $\overrightarrow{AB}$. [1 mark]

b. Find the distance AB. [2 marks]
c. Find the position vector of the midpoint of AB. [2 marks]
d. The point P lies on AB with $AP : PB = 2 : 3$. Find the coordinates of P. [3 marks]

Question 4

Question 4 – Collinear points *(Section A · Skills practice)*

The points $A(1, 0, 2)$, $B(4, 3, 8)$ and $C(6, 5, 12)$ are given.

a. Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$. [2 marks]

b. Show that A, B and C are collinear. [2 marks]

c. Find the ratio $AB : BC$. [1 mark]

Question 5

Question 5 – Parallel vectors *(Section A · Skills practice)*

Find the values of p and q for which each pair of vectors is parallel.

a. $p\mathbf{i} + q\mathbf{j} + 12\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. [3 marks]
b. $6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$ and $4\mathbf{i} + p\mathbf{j} + q\mathbf{k}$. [3 marks]

Question 6

Question 6 – Completing a parallelogram *(Section A · Skills practice)*

$ABCD$ is a parallelogram with $A(1, 2, 0)$, $B(4, 3, 1)$ and $C(6, 7, 5)$.

a. Find the coordinates of D. [3 marks]
b. Show that the diagonals AC and BD have the same midpoint. [2 marks]

Question 7

Question 7 – Flight of a drone *(Section B · Real-life practice)*

A drone is launched from a rooftop. At time t seconds its position vector relative to a fixed origin is $\mathbf{r} = (5 + 2t)\mathbf{i} + (3 - t)\mathbf{j} + (10 + 2t)\mathbf{k}$ metres, where $\mathbf{k}$ is vertically upwards.

a. Find the position of the drone at launch and its height above the origin. [2 marks]
b. Show that the drone flies at constant velocity and find its speed. [3 marks]
c. Find its position when $t = 10$. [1 mark]
d. The drone has to stay below a height of 40 m. Find the time at which it reaches this height. [2 marks]

Question 8

Question 8 – A diagonal in a classroom *(Section B · Real-life practice)*

A rectangular classroom is 8 m long, 6 m wide and 3 m high. A projector cable runs in a straight line from a corner of the floor to the opposite corner of the ceiling.

a. Taking one floor corner as the origin, write down the coordinates of the opposite ceiling corner. [1 mark]
b. Find the length of the cable. [2 marks]
c. Find the angle the cable makes with the floor. [3 marks]

Question 9

Question 9 – Aircraft paths *(Section B · Real-life practice)*

Two light aircraft fly at constant velocity. Relative to a control tower at the origin, with $\mathbf{k}$ vertically upwards and distances in km, their position vectors at time t minutes are $\mathbf{r}_1 = (1+2t)\mathbf{i} + (2+t)\mathbf{j} + (3-t)\mathbf{k}$ and $\mathbf{r}_2 = (11-3t)\mathbf{i} + 2t\mathbf{j} + (5-2t)\mathbf{k}$.

a. Find the speed of each aircraft in km per minute. [3 marks]
b. Show that the aircraft collide, and find the position of the collision. [5 marks]
c. A third aircraft has position $\mathbf{r}_3 = (11-3t)\mathbf{i} + 2t\mathbf{j} + (6-2t)\mathbf{k}$. Show that it does not collide with the first aircraft. [3 marks]

Question 10

Question 10 – Pitching a tent *(Section B · Real-life practice)*

A tent has a rectangular base $ABCD$ with $A(0,0,0)$, $B(4,0,0)$, $C(4,3,0)$ and $D(0,3,0)$ in metres. The apex E is at $(2, 1.5, 2.5)$, directly above the centre of the base.

a. Find $\overrightarrow{AE}$ and show that $AE = \dfrac{5\sqrt{2}}{2}$ m. [3 marks]
b. Show that all four sloping edges have the same length. [3 marks]
c. Find the angle between a sloping edge and the base. [2 marks]

Question 11

Question 11 – The zip wire *(Section B · Real-life practice)*

A zip wire at an adventure park runs from the top of a tower $A(0,0,24)$ to a landing point $B(80,60,0)$, with all distances in metres and $\mathbf{k}$ vertically upwards.

a. Find $\overrightarrow{AB}$ and the length of the wire. [3 marks]
b. Find the angle the wire makes with the horizontal. [3 marks]
c. A safety net is at the point on the wire which is 12 m above the ground. Find its coordinates. [2 marks]

Question 12

Question 12 – Balanced forces on a lamp *(Section B · Real-life practice)*

A lamp hangs in equilibrium under three forces $\mathbf{F}_1 = (4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ N, $\mathbf{F}_2 = (-\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$ N and $\mathbf{F}_3$.

a. Write down the condition for equilibrium and hence find $\mathbf{F}_3$. [3 marks]
b. Find the magnitude of $\mathbf{F}_3$. [2 marks]
c. $\mathbf{F}_3$ is removed. The lamp has mass 2 kg. Find the magnitude of its acceleration. [3 marks]

Question 13

Question 13 – A pyramid roof *(Section B · Real-life practice)*

A square pyramid roof has base $ABCD$ with $A(0,0,0)$, $B(6,0,0)$, $C(6,6,0)$, $D(0,6,0)$ and apex $V(3,3,4)$ in metres.

a. Find $\overrightarrow{VA}$ and the length of the edge VA. [3 marks]

b. Find the angle between the edge VA and the base. [3 marks]

c. Find the coordinates of the midpoint M of BC and the length VM. [3 marks]

Question 14

Question 14 – A radio relay mast *(Section B · Real-life practice)*

A telecommunications company wants to place a relay mast on the straight line joining two towers $A(2,-3,1)$ and $B(10,5,9)$ (coordinates in km, with the third coordinate the height above sea level). The mast P should be three quarters of the way from A to B.

a. Find the coordinates of P. [3 marks]

b. Find the distance AB. [2 marks]

c. Find the distance from P to B. [1 mark]

d. Find the height gain per kilometre of horizontal distance along the line. [2 marks]

Question 15

Question 15 – The roof truss *(Section C · Problem solving)*

Three joints of a roof truss are at $A(2,1,3)$, $B(5,5,15)$ and $C(2,1,8)$ with all lengths in metres.

a. Find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. [3 marks]

b. Find the length of each side of triangle ABC. [3 marks]

c. Use the cosine rule to find angle BAC. [3 marks]

d. Find the area of the triangle and the perpendicular distance from B to AC. [3 marks]

Question 16

Question 16 – Closest approach *(Section C · Problem solving)*

Two drones fly at constant velocity. Relative to an origin O with distances in km, their position vectors after t minutes are $\mathbf{r}_1 = (1+2t)\mathbf{i} + (2+t)\mathbf{j} + (3-t)\mathbf{k}$ and $\mathbf{r}_3 = (11-3t)\mathbf{i} + 2t\mathbf{j} + (6-2t)\mathbf{k}$.

a. Show that the vector from drone 1 to drone 3 is $(10-5t)\mathbf{i} + (t-2)\mathbf{j} + (3-t)\mathbf{k}$. [2 marks]

b. Show that the square of the distance between the drones is $d^2 = 27t^2 - 110t + 113$. [3 marks]
c. Find the value of t for which the drones are closest together and find the minimum distance. [5 marks]

Question 17

Question 17 – The tetrahedron *(Section C · Problem solving)*

$OABC$ is a tetrahedron with $O(0, 0, 0)$, $A(4, 0, 0)$, $B(0, 6, 0)$ and $C(0, 0, 8)$. M is the midpoint of AB and G is the point on CM with $CG : GM = 2 : 1$.

a. Find the coordinates of M. [1 mark]
b. Find the coordinates of G. [3 marks]
c. Show that $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ and describe the point G. [3 marks]
d. Find the lengths OG and CM. [3 marks]

Question 18

Question 18 – A cuboid shipping box *(Section C · Problem solving)*

A shipping box is a cuboid 8 cm by 6 cm by 5 cm. A thin rod rests inside the box along the longest possible straight line, from a bottom corner to the opposite top corner.

Edge	Length (cm)
Length	8
Width	6
Height	5

a. Find the length of the longest rod that fits in the box. [3 marks]
b. Find the angle the rod makes with the base. [3 marks]
c. Find the angle the rod makes with the 8 cm edge. [3 marks]
d. A rod of length 11.5 cm is posted into the box. Explain whether it fits. [1 mark]

Question 19

Question 19 – Passing a radio mast *(Section C · Problem solving)*

A weather balloon is tracked from a radio mast at the origin. Its position vector after t hours is $\mathbf{r} = (4 - t)\mathbf{i} + (2 + 2t)\mathbf{j} + (6 - 2t)\mathbf{k}$ km, with $\mathbf{k}$ vertically upwards.

a. Show that the balloon moves with constant velocity and find its speed. [3 marks]
b. Show that the distance d from the mast satisfies $d^2 = 9t^2 - 24t + 56$. [3 marks]
c. Find the least distance between the balloon and the mast, and the time at which it occurs. [4 marks]
d. At what time does the balloon reach the ground ($z = 0$)? Is it still within 8 km of the mast then? [3 marks]

Question 20

Question 20 – Challenge: diagonals of a parallelogram *(Section D · Challenge)*

$ABCD$ is a parallelogram.

a. Prove, using position vectors, that the diagonals AC and BD bisect each other. [4 marks]
b. $A(1, 2, 3)$, $B(4, 6, 3)$, $C(7, 6, 9)$. Find D and confirm your result from part (a) numerically. [3 marks]

Question 21

Question 21 – Challenge: a right angle *(Section D · Challenge)*

O is the origin, B is $(2, 2, 1)$ and C is $(-2, 1, \lambda)$.

a. Show that $\angle BOC = 90^\circ$ when $\lambda = 2$. [3 marks]
b. Find the value of λ for which $\angle BOC = 90^\circ$ using Pythagoras, and verify that $\lambda = 2$ is the only solution. [4 marks]
c. Hence find the area of triangle OBC. [2 marks]