Answers - Vectors in three dimensions

Year: Year 13

Answer to Question 1

Method: work component by component. For $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the magnitude is $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$, and the vector from A to B is [redacted].

a. $\mathbf{a} + \mathbf{b} = (3 - 1)\mathbf{i} + (-2 + 4)\mathbf{j} + (5 + 2)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$.
b. $3\mathbf{a} = 9\mathbf{i} - 6\mathbf{j} + 15\mathbf{k}$ and $2\mathbf{b} = -2\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$. Subtract: $3\mathbf{a} - 2\mathbf{b} = 11\mathbf{i} - 14\mathbf{j} + 11\mathbf{k}$.
c. $|\mathbf{a}| = \sqrt{9 + 4 + 25} = \sqrt{38} = 6.16$. $|\mathbf{a} + \mathbf{b}| = \sqrt{4 + 4 + 49} = \sqrt{57} = 7.55$.

Answer to Question 2

Method: work component by component. For $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the magnitude is $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$, and the vector from A to B is [redacted].

a. Square each component, add and take the square root.
$|\mathbf{v}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$. So $\mathbf{v}$ has magnitude 7.
b. A unit vector has magnitude 1, so divide $\mathbf{v}$ by its magnitude. $\hat{\mathbf{v}} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$.
Check: $\sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$.
c. Multiply the unit vector by 21. $21\hat{\mathbf{v}} = 3(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = 6\mathbf{i} + 9\mathbf{j} + 18\mathbf{k}$. Its magnitude is $3 \times 7 = 21$.
d. Write the magnitude condition and square both sides. $16 + \lambda^2 + 64 = 144$, so $\lambda^2 = 64$ and $\lambda = \pm 8$. So $\lambda = 8$ or $\lambda = -8$.

Answer to Question 3

Method: work component by component. For $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the magnitude is $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$, and the vector from A to B is [redacted].

a. Subtract the position vector of A from that of B. [redacted].
b. The distance is the magnitude of [redacted]. $AB = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$. So A and B are 13 units apart.
c. The midpoint has position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.
$\frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(7\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) = 3.5\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. So the midpoint is $(3.5, 1, -2)$.

d. $AP : PB = 2 : 3$ means P is $\frac{2}{5}$ of the way from A to B.
$\mathbf{p} = \mathbf{a} + \frac{2}{5}(\mathbf{b} - \mathbf{a}) = (2, -1, 4) + (1.2, 1.6, -4.8) = (3.2, 0.6, -0.8)$. So P is the point $(3.2, 0.6, -0.8)$.

Answer to Question 4

Method: two vectors are parallel when one is a scalar multiple of the other. Points A, B, C are collinear when [redacted] and [redacted] are parallel, because they share the point B.

a. [redacted] and [redacted].

b. [redacted], so the vectors are parallel. They share the point B, so A, B, C lie on one straight line.

c. $AB : BC = 3 : 2$.

Answer to Question 5

Method: two vectors are parallel when one is a scalar multiple of the other. Points A, B, C are collinear when [redacted] and [redacted] are parallel, because they share the point B.

a. Use the component that contains no unknown to find the scale factor. The $\mathbf{k}$ components give the scale factor $12 \div 4 = 3$, so $p = 3 \times 2 = 6$ and $q = 3 \times (-3) = -9$. So $p = 6$ and $q = -9$.

b. Again use the component with no unknown, this time the $\mathbf{i}$ component. The scale factor from the $\mathbf{i}$ components is $4 \div 6 = \frac{2}{3}$. So $p = \frac{2}{3} \times 9 = 6$ and $q = \frac{2}{3} \times (-3) = -2$. Check: $4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k} = \frac{2}{3}(6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$.

Answer to Question 6

Method: in a parallelogram $ABCD$, opposite sides are equal and parallel, so [redacted] and [redacted]. The diagonals bisect each other.

a. In a parallelogram [redacted]. So [redacted]. So D is the point $(3, 6, 4)$. Check: [redacted].

b. Find each midpoint with $\frac{1}{2}(\mathbf{p}+\mathbf{q})$. Midpoint of $AC = \frac{1}{2}(7,9,5) = (3.5, 4.5, 2.5)$. Midpoint of $BD = \frac{1}{2}(7,9,5) = (3.5, 4.5, 2.5)$. They are equal. So the diagonals bisect each other, as they should in a parallelogram.

Answer to Question 7

a. At $t = 0$, $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} + 10\mathbf{k}$, so the drone starts at $(5, 3, 10)$, 10 m above the origin.
b. $\dfrac{d\mathbf{r}}{dt} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, which does not depend on t: constant velocity. Speed $= \sqrt{4+1+4} = 3$ m s^{-1}.
c. $\mathbf{r} = 25\mathbf{i} - 7\mathbf{j} + 30\mathbf{k}$, i.e. $(25, -7, 30)$.
d. $10 + 2t = 40$ gives $t = 15$ s.

Answer to Question 8

Method: in a 3D shape, set up coordinates, find the vector along each edge, then use magnitudes for lengths. For the angle between a line and a horizontal plane, use a right-angled triangle whose sides are the height and the horizontal distance.

a. Taking axes along the room: $(8, 6, 3)$.
b. Length $= \sqrt{8^2 + 6^2 + 3^2} = \sqrt{64 + 36 + 9} = \sqrt{109} = 10.4$ m.
c. The projection of the cable on the floor is the floor diagonal $\sqrt{64 + 36} = 10$ m. So $\tan\theta = \dfrac{3}{10}$ and $\theta = 16.7°$.

Answer to Question 9

a. $\mathbf{v}_1 = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ with speed $\sqrt{6} = 2.45$ km/min. $\mathbf{v}_2 = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ with speed $\sqrt{17} = 4.12$ km/min.
b. Equate components: $\mathbf{i}$: $1 + 2t = 11 - 3t$ so $t = 2$. $\mathbf{j}$: $2 + t = 2t$ so $t = 2$. $\mathbf{k}$: $3 - t = 5 - 2t$ so $t = 2$. All three give the same t, so they collide at $t = 2$ at $(5, 4, 1)$.
c. Equating $\mathbf{k}$ components: $3 - t = 6 - 2t$ gives $t = 3$, but the $\mathbf{i}$ components give $t = 2$. The values differ, so the aircraft are never at the same point at the same time.

Answer to Question 10

Method: in a 3D shape, set up coordinates, find the vector along each edge, then use magnitudes for lengths. For the angle between a line and a horizontal plane, use a right-angled triangle whose sides are the height and the horizontal distance.

a. . $AE = \sqrt{4 + 2.25 + 6.25} = \sqrt{12.5} = \dfrac{5\sqrt{2}}{2} = 3.54$ m.

b. , , . Each has magnitude $\sqrt{4 + 2.25 + 6.25} = \sqrt{12.5}$.

c. Half the base diagonal is $\frac{5}{2} = 2.5$ m and the height is 2.5 m, so $\tan\theta = 1$ and $\theta = 45°$.

Answer to Question 11

Method: in a 3D shape, set up coordinates, find the vector along each edge, then use magnitudes for lengths. For the angle between a line and a horizontal plane, use a right-angled triangle whose sides are the height and the horizontal distance.

a. Subtract the coordinates of A from those of B, then find the magnitude. . Length $= \sqrt{6400 + 3600 + 576} = \sqrt{10576} = 102.8$ m. So the wire is about 103 m long.

b. Use the vertical drop and the horizontal distance. Horizontal distance $= \sqrt{80^2 + 60^2} = 100$ m. $\tan\theta = \dfrac{24}{100}$, so $\theta = 13.5°$. So the wire slopes at about $13.5°$.

c. The height falls steadily from 24 m to 0 m along the wire. 12 m is halfway down, so the point is the midpoint: $(40, 30, 12)$. So the net is at $(40, 30, 12)$.

Answer to Question 12

Method: a particle is in equilibrium when the vector sum of the forces is zero. If one force is removed, the resultant of the others equals minus that force, and Newton's second law $\mathbf{F} = m\mathbf{a}$ gives the acceleration.

a. Equilibrium: $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$. $\mathbf{F}_1 + \mathbf{F}_2 = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, so $\mathbf{F}_3 = -3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$ N.

b. $|\mathbf{F}_3| = \sqrt{9 + 9 + 9} = 3\sqrt{3} = 5.20$ N.

c. The resultant is $\mathbf{F}_1 + \mathbf{F}_2 = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ with magnitude $3\sqrt{3} = 5.196$ N. $a = \dfrac{5.196}{2} = 2.60$ m s^{-2}.

Answer to Question 13

Method: in a 3D shape, set up coordinates, find the vector along each edge, then use magnitudes for lengths. For the angle between a line and a horizontal plane, use a right-angled triangle whose sides are the height and the horizontal distance.

a. Subtract the coordinates of V from those of A. .

$VA = \sqrt{9+9+16} = \sqrt{34} = 5.83$ m. So each sloping edge is about 5.83 m long.

b. The centre of the base is $(3, 3, 0)$, directly below V. Horizontal distance from the centre to A is $\sqrt{18} = 4.243$. $\tan\theta = \dfrac{4}{4.243}$, so $\theta = 43.3°$. So the edge makes about $43.3°$ with the base.

c. Use the midpoint formula, then the magnitude of . $M = (6, 3, 0)$.

, so $VM = \sqrt{9+16} = 5$ m. So the slant height of a face is 5 m.

Answer to Question 14

Method: work component by component. For $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the magnitude is $|\mathbf{v}| = \sqrt{x^2+y^2+z^2}$, and the vector from A to B is .

a. P is three quarters of the way, so add 4 to $\mathbf{a}$. , so $\mathbf{p} = \mathbf{a} + \frac{3}{4}(8, 8, 8) = (2, -3, 1) + (6, 6, 6) = (8, 3, 7)$. So the mast is at $(8, 3, 7)$.

b. $AB = \sqrt{64+64+64} = 8\sqrt{3} = 13.9$ km. So the towers are about 13.9 km apart.

c. P is three quarters of the way, so PB is the remaining quarter. $PB = \frac{1}{4}AB = 2\sqrt{3} = 3.46$ km.

d. Split into its horizontal part $(8, 8)$ and its vertical part 8. Horizontal distance is $\sqrt{64+64} = 8\sqrt{2}$ and height gain is 8, so the gradient is $\dfrac{8}{8\sqrt{2}} = 0.707$ km per km. So the line rises about 0.707 km for each kilometre across.

Answer to Question 15

a. , , .

b. $AB = \sqrt{9+16+144} = 13$. $AC = 5$. $BC = \sqrt{9+16+49} = \sqrt{74} = 8.60$.

c. $\cos A = \dfrac{13^2 + 5^2 - 74}{2 \times 13 \times 5} = \dfrac{120}{130} = 0.923$, so $A = 22.6°$.

d. Area $= \frac{1}{2} \times 13 \times 5 \times \sin 22.6° = 12.5$ m^2. Perpendicular distance from B to AC is $\dfrac{2 \times 12.5}{5} = 5$ m (the $\mathbf{i}$ and $\mathbf{j}$ components, $\sqrt{9+16}$, which agrees).

Answer to Question 16

Method: work component by component. For $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the magnitude is $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$, and the vector from A to B is .

a. Subtract $\mathbf{r}_1$ from $\mathbf{r}_3$ component by component.
$\mathbf{r}_3 - \mathbf{r}_1 = (11 - 3t - 1 - 2t)\mathbf{i} + (2t - 2 - t)\mathbf{j} + (6 - 2t - 3 + t)\mathbf{k} = (10 - 5t)\mathbf{i} + (t - 2)\mathbf{j} + (3 - t)\mathbf{k}$. This is the required vector.

b. Square each component and add.
$d^2 = (10 - 5t)^2 + (t - 2)^2 + (3 - t)^2 = (25t^2 - 100t + 100) + (t^2 - 4t + 4) + (t^2 - 6t + 9) = 27t^2 - 110t + 113$. This is the required expression.

c. Minimising d^2 also minimises d, so differentiate d^2 with respect to t.
$\frac{d(d^2)}{dt} = 54t - 110 = 0$, so $t = \frac{55}{27} = 2.04$ minutes. Then
$d^2 = 27\left(\frac{55}{27}\right)^2 - 110 \times \frac{55}{27} + 113 = 113 - \frac{3025}{27} = 0.963$, so $d = 0.981$ km. So the drones are closest after about 2.04 minutes, at a distance of about 0.98 km.

Answer to Question 17

Method: work component by component. For $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the magnitude is $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$, and the vector from A to B is .

a. $M = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = (2, 3, 0)$.

b. G is two thirds of the way from C to M. .

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c. $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{3}(4, 6, 8) = \left(\frac{4}{3}, 2, \frac{8}{3}\right) = \mathbf{g}$. G is the centroid of triangle ABC. The centroid is where the three medians of the triangle meet.

d. Find the magnitudes of $\mathbf{g}$ and . $OG = \sqrt{\frac{16}{9} + 4 + \frac{64}{9}} = \sqrt{\frac{116}{9}} = 3.59$.
$CM = \sqrt{4 + 9 + 64} = \sqrt{77} = 8.77$. So $OG \approx 3.59$ and $CM \approx 8.77$.

Answer to Question 18

Method: in a 3D shape, set up coordinates, find the vector along each edge, then use magnitudes for lengths. For the angle between a line and a horizontal plane, use a right-angled triangle whose sides are the height and the horizontal distance.

a. Space diagonal $= \sqrt{64 + 36 + 25} = \sqrt{125} = 11.18$ cm.

b. Base diagonal $= \sqrt{100} = 10$. $\tan\theta = \dfrac{5}{10} = 0.5$, so $\theta = 26.6°$.

c. $\cos\phi = \dfrac{8}{11.18} = 0.7155$, so $\phi = 44.3°$.

d. 11.5 > 11.18, so it does **not** fit (it would stick out of the box).

Answer to Question 19

a. $\dfrac{d\mathbf{r}}{dt} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ (constant). Speed $= \sqrt{1+4+4} = 3$ km/h.

b. $d^2 = (4-t)^2 + (2+2t)^2 + (6-2t)^2 = (16 - 8t + t^2) + (4 + 8t + 4t^2) + (36 - 24t + 4t^2) = 9t^2 - 24t + 56$.

c. $\dfrac{d(d^2)}{dt} = 18t - 24 = 0$ gives $t = \frac{4}{3}$ h.
$d^2 = 9 \times \frac{16}{9} - 24 \times \frac{4}{3} + 56 = 16 - 32 + 56 = 40$, so $d_{min} = \sqrt{40} = 6.32$ km.

d. $6 - 2t = 0$ gives $t = 3$ h. $d^2 = 81 - 72 + 56 = 65$, so $d = 8.06$ km, which is **more** than 8 km, so the balloon is not within 8 km.

Answer to Question 20

a. In a parallelogram , so $\mathbf{d} - \mathbf{a} = \mathbf{c} - \mathbf{b}$ and $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$. The midpoint of BD is $\frac{1}{2}(\mathbf{b} + \mathbf{d}) = \frac{1}{2}(\mathbf{b} + \mathbf{a} + \mathbf{c} - \mathbf{b}) = \frac{1}{2}(\mathbf{a} + \mathbf{c})$, the midpoint of AC. The midpoints coincide, so the diagonals bisect each other.

b. $\mathbf{d} = (1,2,3) + (7,6,9) - (4,6,3) = (4,2,9)$. Midpoint of $AC = (4,4,6)$. Midpoint of $BD = \frac{1}{2}(4+4, 6+2, 3+9) = (4,4,6)$.

Answer to Question 21

a. $OB^2 = 9$, $OC^2 = 4 + 1 + 4 = 9$, $BC^2 = 16 + 1 + 1 = 18$. Since $OB^2 + OC^2 = 18 = BC^2$, the angle at O is a right angle.

b. In general $OB^2 = 9$, $OC^2 = 5 + \lambda^2$, $BC^2 = 16 + 1 + (\lambda - 1)^2 = 18 - 2\lambda + \lambda^2$. Setting $OB^2 + OC^2 = BC^2$: $14 + \lambda^2 = 18 - 2\lambda + \lambda^2$, so $2\lambda = 4$ and $\lambda = 2$ (the quadratic terms cancel, leaving a linear equation, so there is one solution).

c. Area $= \frac{1}{2} \times OB \times OC = \frac{1}{2} \times 3 \times 3 = 4.5$.