Test answers - Proof by contradiction

Year: Year 13 · Duration: 30 minutes · /20

Answer to Question 1 (4 marks)

a. "n is an odd integer" (or "n is not even").
b. Assume n^2 is even and n is odd, so $n = 2k + 1$. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. This contradicts n^2 being even [2 marks]. Hence n is even [1 mark]. ■

The proof must start by assuming the opposite of the conclusion, here that n is odd, while keeping the hypothesis that n^2 is even. Writing the odd number as $2k + 1$ with k an integer is essential; a numerical example such as $3^2 = 9$ proves nothing. The final mark needs a concluding statement.

Answer to Question 2 (4 marks)

Assume $\sqrt{5} = \dfrac{a}{b}$ where a, b are integers with no common factor [1 mark]. Then $a^2 = 5b^2$, so $5 \mid a^2$ and therefore $5 \mid a$ [1 mark]. Write $a = 5k$: $25k^2 = 5b^2$, so $b^2 = 5k^2$, so $5 \mid b$ [1 mark]. Both are divisible by 5, which contradicts the assumption that the fraction was in lowest terms, so $\sqrt{5}$ is irrational [1 mark]. ■

The step $5 \mid a^2 \Rightarrow 5 \mid a$ is valid because 5 is prime.

Answer to Question 3 (4 marks)

Assume integers m, n exist with $12m + 18n = 5$ [1 mark]. Then $6(2m + 3n) = 5$ [1 mark], so $2m + 3n = \dfrac{5}{6}$ [1 mark]. But $2m + 3n$ is an integer, and $\dfrac{5}{6}$ is not. This is a contradiction, so no such integers exist [1 mark]. ■

Equivalently, the left-hand side is a multiple of 6 for all integers, while 5 is not a multiple of 6. The key idea is to factorise the highest common factor of 12 and 18; dividing by 2 only, to get $6m + 9n = \frac{5}{2}$, works equally well since the left side is then an integer.

Answer to Question 4 (4 marks)

a. $N = p_1 p_2 \cdots p_k + 1$.
b. Each p_i divides $p_1 p_2 \cdots p_k$, so dividing N by p_i leaves remainder 1. So no p_i divides N [2 marks].
c. $N > 1$ so N has a prime factor (it is prime or a product of primes). That prime factor is not on the list, but the list was supposed to contain all primes.

Contradiction. ■

Note that N itself need not be prime: $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$.

Answer to Question 5 (4 marks)

a. Assume $xy = \frac{p}{q}$ with integers p, q, $q \neq 0$, and $x = \frac{r}{s}$ with $r \neq 0$. Then $y = \frac{ps}{qr}$, a ratio of integers, so y is rational. This contradicts y being irrational. Hence xy is irrational. ■

b. $\frac{\sqrt{2}}{3} = \frac{1}{3} \times \sqrt{2}$ with $\frac{1}{3}$ rational and non-zero and $\sqrt{2}$ irrational, so it is **irrational**, by part (a).

The condition $x \neq 0$ is needed so that $r \neq 0$ and we may divide by x; with $x = 0$, $xy = 0$ is rational.