Worksheet - Proof by contradiction

Year: Year 13 · 20

Show your working.

Question 1

Question 1 – Writing negations *(Section A · Skills practice)*

The first step in a proof by contradiction is to write down the opposite of the statement. Write the negation of each statement, in good English.

a. n is a multiple of 3. [1 mark]
b. There are infinitely many prime numbers. [1 mark]
c. For all real x, $x^2 \geq 0$. [2 marks]
d. $\sqrt{x}$ is irrational. [1 mark]
e. There exist integers m and n such that $6m + 9n = 1$. [2 marks]

Question 2

Question 2 – Squares and parity *(Section A · Skills practice)*

Use proof by contradiction to prove each statement, where n is an integer.

a. If n^2 is even, then n is even. [3 marks]
b. If n^2 is divisible by 3, then n is divisible by 3. [4 marks]

Question 3

Question 3 – Irrational times rational *(Section A · Skills practice)*

You may assume that $\sqrt{2}$ is irrational.

a. Prove by contradiction that if x is a non-zero rational number and y is irrational, then xy is irrational. [4 marks]
b. Hence prove that $3\sqrt{2}$ is irrational. [1 mark]
c. Prove that $1 + \sqrt{2}$ is irrational. [3 marks]

Question 4

Question 4 – Irrationality of the square root of 3 *(Section A · Skills practice)*

Prove that $\sqrt{3}$ is irrational by completing the following steps.

a. Assume $\sqrt{3} = \dfrac{a}{b}$ in lowest terms and show that $a^2 = 3b^2$. [2 marks]
b. Explain why a is divisible by 3. [2 marks]

c. Write $a = 3k$ and deduce that b is also divisible by 3. [2 marks]
d. State the contradiction and conclude. [2 marks]

Question 5

Question 5 – No integer solutions *(Section A · Skills practice)*

Prove by contradiction that there are no integers m and n satisfying each equation.

a. $6m + 9n = 1$ [3 marks]
b. $10m + 15n = 3$ [3 marks]
c. $x^2 - 4y = 2$, where x and y are integers [4 marks]

Question 6

Question 6 – Spotting the flaw *(Section A · Skills practice)*

A student writes the following "proof" that $\sqrt{2}$ is irrational.

"Assume $\sqrt{2} = \frac{a}{b}$. Then $a^2 = 2b^2$, so a is even and $a = 2k$. Then $4k^2 = 2b^2$ and $b^2 = 2k^2$, so b is even. Therefore $\sqrt{2}$ is irrational."

a. Identify two things the student has not stated. [2 marks]
b. Explain why, without the missing statement, there is no contradiction. [2 marks]
c. Rewrite the proof so that it would gain full marks. [4 marks]

Question 7

Question 7 – The diagonal of a square tile *(Section B · Real-life practice)*

A tiler cuts square floor tiles of side 1 unit and wants to measure the diagonal exactly as a fraction of the side, say $\frac{a}{b}$ of a unit with a and b whole numbers of tiny marks on her ruler.

a. Show that the diagonal has length $\sqrt{2}$ using Pythagoras. [1 mark]
b. Prove that no such fraction exists. [5 marks]
c. The tiler measures 1.414 and says the diagonal is $\frac{1414}{1000}$ units. Explain why this must be an approximation. [2 marks]

Question 8

Question 8 – The chessboard and the dominoes *(Section B · Real-life practice)*

A 8×8 chessboard has two opposite corner squares removed. A student claims that the remaining 62 squares can be covered exactly by 31 dominoes, each covering two adjacent squares.

a. How many black and how many white squares remain? [2 marks]
b. Use proof by contradiction to show that the student is wrong. [4 marks]
c. Would the claim be true if two adjacent squares were removed instead? Justify your answer. [2 marks]

Question 9

Question 9 – The five-a-side league *(Section B · Real-life practice)*

A new league has 7 teams. The organiser wants every team to play exactly 3 matches, each against a different opponent, and no pair of teams to meet twice.

a. Assume the schedule is possible. Show that the total number of matches would have to be $\frac{7 \times 3}{2}$. [3 marks]
b. Explain why this is a contradiction. [2 marks]
c. For what values of n teams is it possible for every team to play exactly 3 matches against different opponents? [3 marks]

Question 10

Question 10 – Sheets of paper *(Section B · Real-life practice)*

A4 paper is cut so that the longer side divided by the shorter side is $\sqrt{2}$, which keeps the shape when it is folded in half. A printing company states that it measures every sheet in whole millimetres.

a. Prove that no sheet can have an exact ratio of $\sqrt{2}$ with both sides a whole number of millimetres. [5 marks]
b. A4 is $297\ \text{mm} \times 210\ \text{mm}$. Find the ratio to 4 decimal places and compare with $\sqrt{2}$. [2 marks]
c. Explain what this tells you about the real size of sheets. [1 mark]

Question 11

Question 11 – Tokens at the school fair *(Section B · Real-life practice)*

At a school fair, prizes cost a number of points. Points are earned only from 6-point and 9-point tokens. A pupil has a bag of both types and wants a prize costing exactly 50 points.

a. Show that if the pupil uses a tokens of 6 points and b tokens of 9 points, the total is a multiple of 3. [2 marks]
b. Prove by contradiction that a 50-point prize cannot be bought exactly. [3 marks]
c. Which of the prizes costing 48, 51, 52 and 57 points can be bought exactly? [3 marks]

Question 12

Question 12 – A rectangular allotment *(Section B · Real-life practice)*

A gardener wants a rectangular allotment with integer side lengths a and b metres, using exactly 30 m of fencing, so the perimeter is 30 m. She also wants the area to be exactly 57 m^2.

a. Show that $a + b = 15$. [1 mark]
b. Use proof by contradiction to show that the gardener's wish is impossible, even without the integer condition. [5 marks]
c. What is the largest area she could enclose? [2 marks]

Question 13

Question 13 – Why twelve fifths never equal seven octaves *(Section B · Real-life practice)*

In music, an octave multiplies a frequency by 2 and a perfect fifth multiplies it by $\frac{3}{2}$.

Pythagoras wondered whether going up 12 perfect fifths could return exactly to a note some whole number of octaves above the start.

a. Show that this would require $3^{12} = 2^{19}$. [3 marks]
b. Prove by contradiction that no whole number n and positive integer m satisfy $3^m = 2^n$. [3 marks]
c. Calculate 3^{12} and 2^{19} and find their ratio to 4 decimal places. [2 marks]

Question 14

Question 14 – Prime keys in computer security *(Section B · Real-life practice)*

A security engineer assumes that a computer system only ever needs the first six primes, 2, 3, 5, 7, 11 and 13, because these are "all the primes".

a. Calculate $N = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1$. [1 mark]
b. Show that N is not divisible by any of the six primes. [2 marks]
c. Factorise N and explain how this shows there are more primes than the engineer assumed. [3 marks]
d. Explain why this does not prove that N itself is prime. [2 marks]

Question 15

Question 15 – Is log base 2 of 3 rational? *(Section C · Problem solving)*

Musicians approximate the number $\log_2 3 = 1.58496\ldots$ by fractions $\frac{n}{m}$, because $3^m = 2^n$ would make twelve fifths fit exactly in octaves. The table shows successive attempts.

Fraction $\frac{n}{m}$	m	n	3^m	2^n	Ratio $3^m : 2^n$
$\frac{3}{2}$	2	3	9	8	1.1250
$\frac{8}{5}$	5	8	243	256	0.9492
$\frac{19}{12}$	12	19	531 441	524 288	1.0136

a. Suppose $\log_2 3 = \frac{p}{q}$ with p, q positive integers. Show that $3^q = 2^p$. [2 marks]
b. Prove that this is impossible, and so that $\log_2 3$ is irrational. [4 marks]
c. Verify the last row of the table. [2 marks]
d. Explain why the fractions in the table can never be exact, however large m is. [2 marks]
e. Which of the fractions $\frac{3}{2}$, $\frac{8}{5}$, $\frac{19}{12}$ is the best approximation to $\log_2 3$? Justify. [2 marks]

Question 16

Question 16 – Sums of square roots *(Section C · Problem solving)*

You may use the fact that $\sqrt{2}$ is irrational.

a. Prove that $\sqrt{3}$ is irrational. [5 marks]
b. Assume that $\sqrt{2} + \sqrt{3} = r$ where r is rational. Show that $\sqrt{2} = \frac{r^2 - 1}{2r}$. [3 marks]
c. Hence prove that $\sqrt{2} + \sqrt{3}$ is irrational. [2 marks]
d. Evaluate $\sqrt{2} + \sqrt{3}$ to 4 decimal places and explain why this decimal cannot prove its irrationality. [2 marks]

Question 17

Question 17 – A cubic with no rational roots *(Section C · Problem solving)*

\($f(x) = x^3 + x + 1$.\)

x	-2	-1	0	1
$f(x)$	-9	-1	1	3

a. Use the table to show that $f(x) = 0$ has a real root between -1 and 0. [2 marks]
b. Assume that $\frac{p}{q}$ is a rational root in lowest terms, with $q > 0$. Show that $p^3 + pq^2 + q^3 = 0$. [2 marks]
c. By considering whether p and q are odd or even, obtain a contradiction. [5 marks]
d. Find the root to 3 decimal places and say what your answers show. [3 marks]

Question 18

Question 18 – Euclid's argument in action *(Section C · Problem solving)*

Euclid proved that there are infinitely many primes using the number $N = p_1 p_2 \cdots p_k + 1$. The table shows N for the first few primes.

Primes used	Product	$N =$ product $+1$	Factorisation of N
2	2	3	3 (prime)
2, 3	6	7	7 (prime)
2, 3, 5	30	31	31 (prime)
2, 3, 5, 7	210	211	211 (prime)
2, 3, 5, 7, 11	2310	2311	2311 (prime)
2, 3, 5, 7, 11, 13	30030	30031	59×509

a. Confirm the last row and explain why it does not disprove Euclid's argument. [3 marks]
b. Prove by contradiction that there is no largest prime, by considering $P! + 1$ where P is the supposed largest prime. [5 marks]
c. Use your proof to explain why there is always a prime between P and $P! + 1$. [2 marks]
d. Show that $5! + 1 = 121$ is not prime and find its prime factorisation. [2 marks]

Question 19

Question 19 – The largest possible field *(Section C · Problem solving)*

A farmer has 40 m of fencing and wants a rectangle of sides a and b metres to enclose the largest possible area. A sample of possible rectangles is shown.

a (m)	b (m)	Area (m^2)
10	10	100
5	15	75
8	12	96
9	11	99

a. Show that $a + b = 20$ and check that the areas in the table are correct. [2 marks]
b. Prove by contradiction that the area can never exceed 100 m^2. [5 marks]
c. State when the area equals 100 m^2. [1 mark]
d. Use the same method to prove that for positive a, b: $\dfrac{a+b}{2} \geq \sqrt{ab}$. [4 marks]

Question 20

Question 20 – Square roots of primes and products *(Section D · Challenge)*

Let p be a prime number.

a. Prove that if $p \mid n^2$ then $p \mid n$. (Hint: use the fact that if a prime divides a product of two integers, it divides at least one of them.) [3 marks]
b. Prove that $\sqrt{p}$ is irrational. [4 marks]
c. Prove that $\sqrt{6}$ is irrational. [4 marks]
d. The proof for $\sqrt{p}$ fails for $\sqrt{4}$. Explain exactly which step fails. [2 marks]