Lesson - Projectiles and variable acceleration

Year: Year 13

A **projectile** is a particle moving freely under gravity alone. This chapter models projectile motion, then uses calculus for motion with variable acceleration.

Starter activity

Throw a ball to a friend and notice its path. At which point is it moving most slowly? Explain why a ball thrown at 45° travels further than one thrown at 15° or 75°. Now, think about a car driving on a motorway: why does the acceleration change with time as the driver presses the accelerator, and what graph would you draw to show this?

Modelling projectile motion

Definition: projectile model

In the usual model we ignore air resistance, treat the object as a particle, assume gravity is constant, acts downwards with $g = 9.8\ \mathrm{m\,s^{-2}}$, and ignore the wind. **Resolve the motion into horizontal and vertical components.** Horizontally there is no acceleration; vertically the acceleration is $-g$, so the suvat equations apply. Projected with speed u at angle θ above the horizontal, the initial components are $u\cos\theta$ and $u\sin\theta$.

The equations of motion

Rule: projectile equations

$x = ut\cos\theta$, $y = ut\sin\theta - \frac{1}{2}gt^2$, $v_x = u\cos\theta$, $v_y = u\sin\theta - gt$. The speed is $\sqrt{v_x^2 + v_y^2}$ at angle $\tan^{-1}\dfrac{|v_y|}{v_x}$ to the horizontal. Eliminating t: $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$, a downward parabola.

Key fact: top of the flight and range

At the highest point $v_y = 0$: time $\dfrac{u\sin\theta}{g}$, height $H = \dfrac{u^2\sin^2\theta}{2g}$, and the particle moves horizontally at $u\cos\theta$. Landing at the launch height: $T = \dfrac{2u\sin\theta}{g}$ and range $R = \dfrac{u^2\sin 2\theta}{g}$, greatest at $\theta = 45^\circ$.

Worked example 1

A particle is projected at $u = 20$ and $\theta = 30^\circ$. Find T, R and H.

$T = 2.04$ s, $R = 35.3$ m and $H = 5.10$ m. For $u = 25$ with $\tan\theta = \frac{3}{4}$, the height at $x = 40$ m is $30 - 19.6 = 10.4$ m.

Method: projectile questions

1. Sketch the launch point, initial velocity and angle.
2. Write two columns, "horizontal" and "vertical", and fill in what is known. Take upwards as positive so g appears as -9.8.
3. The shared quantity is the time t: find it from one column and use it in the other.
4. For speed or direction at a point, find both components and combine them with Pythagoras and a tangent. "At the highest point": $v_y = 0$.
5. Give 3 significant figures, with units.

Worked example 2

A stone is thrown horizontally at $12\ \mathrm{m\,s^{-1}}$ from a cliff 45 m high. Find the distance and the landing speed.

$45 = \frac{1}{2}(9.8)t^2$, so $t = 3.03$ s and the horizontal distance is $12 \times 3.03 = 36.4$ m. At landing $v_y = 29.7$, so the speed is $\sqrt{12^2 + 29.7^2} = 32.0\ \mathrm{m\,s^{-1}}$ at 68.0° below the horizontal.

Worked example 3

A ball hit at 30 m/s lands 60 m away on level ground. Find the angle.

$\sin 2\theta = \dfrac{gR}{u^2} = \dfrac{588}{900} = 0.6533$, so $2\theta = 40.8^\circ$ or 139.2°, and $\theta = 20.4^\circ$ or 69.6°. The two angles add to 90° and give the same range.

Worked example 4

A ball at 20 m/s and 35° meets a wall 30 m away. Find its height there, and its speed at $y = 3$ m for $u = 20$.

$t = \dfrac{30}{20\cos 35^\circ} = 1.83$ s, $y = 11.47(1.83) - 4.9(1.83)^2 = 4.58$ m. Using $v^2 = u^2 - 2gy$: $v = \sqrt{400 - 58.8} = 18.5\ \mathrm{m\,s^{-1}}$.

Watch out

Do not swap sine and cosine when resolving. v_y changes sign at the top, but only the

vertical component is zero there: the speed at the top is $u\cos\theta$, not 0.

Variable acceleration with calculus

Rule: calculus in kinematics

When acceleration is not constant the suvat equations do not apply. Use $v = \dfrac{ds}{dt}$, $a = \dfrac{dv}{dt}$, and in reverse $v = \int a\,dt$, $s = \int v\,dt$. Use the initial conditions to find each constant.

Worked example 5

$a = 6t - 4$, with $v = 2$ and $s = 0$ at $t = 0$. Find v and s.

$v = 3t^2 - 4t + 2$ and $s = t^3 - 2t^2 + 2t$.

Worked example 6

$v = 3t^2 - 2t + 1$. Find a at $t = 2$ and the displacement from $t = 0$ to 2. Then for $v = 4t - t^2$ find when it is at rest, the maximum speed and the distance in the first 4 s.

$a = 6t - 2 = 10$ at $t = 2$; displacement $\int_0^2 v\,dt = 8 - 4 + 2 = 6$ m. For $v = 4t - t^2$: at rest at $t = 0$ and 4; $\dfrac{dv}{dt} = 4 - 2t = 0$ at $t = 2$, maximum $v = 4$; distance $\int_0^4 (4t - t^2)\,dt = \frac{32}{3}$ m.

Displacement versus distance

Definition: displacement and distance

$\int v\,dt$ gives **displacement**, which is signed. The **total distance** needs $v = 0$ found first, then each part integrated separately and taken positive.

Worked example 7

$v = t^2 - 4t + 3$ from $t = 0$ to 3. Find the displacement and the distance.

$v = (t - 1)(t - 3)$. The displacement is 0, but the particle moves forward $\frac{4}{3}$ m then back $\frac{4}{3}$ m, so the distance is $\frac{8}{3}$ m.

Worked example 8

$\mathbf{r} = t^2\mathbf{i} + (2t^3 - 3t)\mathbf{j}$. Find $\mathbf{v}$, $\mathbf{a}$ and the speed at $t = 1$.

$\mathbf{v} = 2t\mathbf{i} + (6t^2 - 3)\mathbf{j}$, $\mathbf{a} = 2\mathbf{i} + 12t\mathbf{j}$. At $t = 1$: $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, speed $\sqrt{13} = 3.61\ \mathrm{m\,s^{-1}}$.

Commenting on the model

Note: limitations

Air resistance would reduce range and height; the object is a particle, so spin and size are ignored; g is constant over small heights; wind is ignored. Say what effect the improvement would have, not just that it exists.

Watch out

Use initial conditions to find constants. Do not use $\int v\,dt$ for the total distance when the particle changes direction.

Note: the path of a projectile



Worked example 9: Speed at a given height

A ball is projected at 20 m s^{-1} at $35°$. Find its speed when it is 5 m above the launch point.

Use conservation of the horizontal component and $v_y^2 = u_y^2 - 2gy$.
$u_x = 20\cos 35° = 16.38$, $u_y = 11.47$. At $y = 5$: $v_y^2 = 131.6 - 98 = 33.6$, so $v_y = \pm 5.80$.
Speed $= \sqrt{16.38^2 + 33.6} = \sqrt{268.3 + 33.6} = 17.4$ m s^{-1}. The same speed applies on the way up and on the way down at this height.

Worked example 10: Does the ball hit the target?

A stone is thrown at 12 m s^{-1} at $30°$ from ground level towards a wall 10 m away. Find the height at which it hits the wall.

$t = \dfrac{10}{12\cos 30°} = 0.962$ s. $y = 12\sin 30° \times 0.962 - 4.9 \times 0.962^2 = 5.77 - 4.53 = 1.24$ m.

Check that the stone has not already landed: the time of flight is $\dfrac{2 \times 6}{9.8} = 1.22$ s, which is longer than 0.962 s, so it reaches the wall at 1.24 m and is going downwards

(since $0.962 > 6/9.8 = 0.61$ s).

Worked example 11: Integrating with a limit

A particle has acceleration $a = 12t - 6$ m s^{-2} and starts at the origin with velocity -2 m s^{-1}. Find the time when it returns to the origin.

$v = 6t^2 - 6t - 2$ (with $C = -2$). $s = 2t^3 - 3t^2 - 2t$ (with $s = 0$ at $t = 0$). Solve $s = 0$: $t(2t^2 - 3t - 2) = 0$, so $t(2t + 1)(t - 2) = 0$ and $t = 2$ s (rejecting $t = 0$ and $t = -\frac{1}{2}$).

Worked example 12: Maximum range from a height

A ball is thrown from a roof, 20 m above level ground, at 15 m s^{-1} horizontally. How far from the building does it land?

$20 = 4.9t^2$, so $t = 2.02$ s. Distance $= 15 \times 2.02 = 30.3$ m.

Worked example 13: Finding when a particle is at its maximum displacement

A particle moves with $v = 6t - 3t^2$ m s^{-1}, starting at the origin. Find the greatest displacement from the origin and when it occurs.

$v = 3t(2 - t) = 0$ at $t = 0$ and $t = 2$. The displacement is $s = 3t^2 - t^3$, so $s(2) = 12 - 8 = 4$ m. After $t = 2$ the velocity is negative, so the particle returns towards the origin and reaches it again when $3t^2 - t^3 = 0$, i.e. $t = 3$ s. The greatest displacement is 4 m at $t = 2$ s.

Worked example 14: Time and height from the quadratic in t

A stone is thrown vertically upwards from the edge of a cliff at 14 m/s. The cliff top is 49 m above the sea. Find the time to reach the sea.

Take upwards as positive: $-49 = 14t - 4.9t^2$, so $4.9t^2 - 14t - 49 = 0$, or $t^2 - \frac{20}{7}t - 10 = 0$. Using the formula, $t = \dfrac{14 + \sqrt{196 + 960.4}}{9.8} = \dfrac{14 + 34.0}{9.8} = 4.90$ s.

The negative root is rejected because the time must be positive. The stone first rises for $\dfrac{14}{9.8} = 1.43$ s.

Worked example 15: Modelling remarks on a projectile

List three assumptions in the basic projectile model and the effect of removing each one for a tennis ball.

(1) No air resistance: with air resistance the ball would travel less far and land at a steeper angle. (2) The ball is a particle, so size and spin are ignored: topspin makes a tennis ball dip faster, backspin makes it float. (3) Constant g and no wind: a following wind would increase the range, a head wind would reduce it. Each comment says what

effect the change would have, which is what examiners look for.

Rule: working with the suvat equations in each direction

For constant acceleration use $v = u + at$, $s = ut + \frac{1}{2}at^2$, $v^2 = u^2 + 2as$ and $s = \frac{1}{2}(u + v)t$ in *each direction separately*. The same time t appears in both. Horizontally $a = 0$, so $s = ut$. Vertically $a = -g$, so $s = ut - \frac{1}{2}gt^2$. For a particle that returns to its launch level, the time up equals the time down and the landing speed equals the launch speed.

Method: projectile or variable acceleration?

1. Is the acceleration constant? For a projectile it is 0 horizontally and $-g$ vertically, so use suvat in each direction separately.
2. If the acceleration is given as a function of t, integrate to find v and s, using initial conditions to find each constant.
3. If a position or velocity is given, differentiate to find v and a.
4. For distance rather than displacement, find where $v = 0$ and integrate between those times, taking each part as positive.

Watch out: signs and speed at the top

Choose upwards as positive and use -9.8 for acceleration. At the top of a projectile path the vertical component is zero but the horizontal component is not, so the speed is $u\cos\theta$. Also, the magnitude of the displacement is not the distance travelled when the particle turns round: check where $v = 0$.

Did you know?

Galileo Galilei showed around 1600 that a projectile follows a parabola, by treating its horizontal and vertical motion separately, an idea that was hard to accept at the time because many people believed in an older theory that a cannonball moved in a straight line until it ran out of impetus. Today the same equations are used to design sports equipment, fountains and the paths of spacecraft after engine cut-off.

Otto's tip

Draw the initial velocity triangle with the angle marked before you write any equation.