Answers - Projectiles and variable acceleration

Year: Year 13

Answer to Question 1

Method: treat the horizontal and vertical motions separately. Horizontally the velocity $u\cos\theta$ is constant; vertically the acceleration is $g = 9.8$ m s^{-2} downwards, so use the suvat equations with initial velocity $u\sin\theta$. The time t links the two motions.

a. Resolve the initial velocity. Horizontal: $24\cos 40^\circ = 18.4$ m s^{-1}. Vertical: $24\sin 40^\circ = 15.4$ m s^{-1}. These are the initial components of the velocity.
b. At the highest point the vertical velocity is zero; use $v = u + at$ vertically. At the top $v_y = 0$: $0 = 15.43 - 9.8t$, so $t = 1.57$ s. So the ball reaches its highest point after about 1.57 s.
c. Use $v^2 = u^2 + 2as$ vertically with $v = 0$. $v^2 = u^2 + 2as$: $0 = 15.43^2 - 2 \times 9.8H$, so $H = \dfrac{238.0}{19.6} = 12.1$ m. So the greatest height is about 12.1 m.

Answer to Question 2

Method: treat the horizontal and vertical motions separately. Horizontally the velocity $u\cos\theta$ is constant; vertically the acceleration is $g = 9.8$ m s^{-2} downwards, so use the suvat equations with initial velocity $u\sin\theta$. The time t links the two motions. Here the ball leaves horizontally, so its initial vertical velocity is zero.

a. Vertically (downwards positive) $s = \frac{1}{2}gt^2$: $1.2 = 4.9t^2$, so $t = \sqrt{0.2449} = 0.495$ s. So the ball is in the air for about 0.495 s.
b. The horizontal velocity stays 3 m s^{-1}. $x = 3 \times 0.495 = 1.48$ m. So the ball lands about 1.48 m from the table.
c. Combine the two components with Pythagoras and the inverse tangent. $v_y = gt = 9.8 \times 0.495 = 4.85$ m s^{-1} downwards, $v_x = 3$. Speed $= \sqrt{9 + 23.5} = 5.70$ m s^{-1}. Angle below the horizontal $= \tan^{-1}\dfrac{4.85}{3} = 58.3^\circ$. So the ball hits the floor at about 5.70 m s^{-1}, 58.3° below the horizontal.

Answer to Question 3

Method: treat the horizontal and vertical motions separately. Horizontally the velocity $u\cos\theta$ is constant; vertically the acceleration is $g = 9.8$ m s^{-2} downwards, so use the suvat equations with initial velocity $u\sin\theta$. The time t links the two motions.

a. The ball lands when the vertical displacement is zero again: $0 = u\sin\theta\, t - 4.9t^2$.

$T = \dfrac{2u\sin\theta}{g} = \dfrac{2 \times 30 \times 0.4226}{9.8} = 2.59$ s. So the ball is in the air for about 2.59 s.

b. Multiply the constant horizontal velocity by the time of flight.
$R = u\cos\theta \times T = 27.19 \times 2.587 = 70.4$ m. (Using $R = \dfrac{u^2\sin 2\theta}{g} = 70.35$ m.) So the range is about 70.4 m.

c. Use $v^2 = u^2 + 2as$ vertically with $v = 0$ at the top.
$H = \dfrac{(u\sin\theta)^2}{2g} = \dfrac{12.68^2}{19.6} = 8.20$ m. So the ball rises about 8.20 m.

Answer to Question 4

Method: when the acceleration depends on t, suvat cannot be used. Differentiate to go from displacement to velocity to acceleration, and integrate (adding a constant found from the initial conditions) to go back.

a. Integrate the acceleration and use $v = 3$ when $t = 0$.
$v = \int (4t - 6)\,dt = 2t^2 - 6t + C$. At $t = 0$, $v = 3$, so $C = 3$ and $v = 2t^2 - 6t + 3$.

b. Integrate the velocity and use $s = 0$ when $t = 0$.
$s = \int (2t^2 - 6t + 3)\,dt = \frac{2}{3}t^3 - 3t^2 + 3t + D$. At $t = 0$, $s = 0$ so $D = 0$ and $s = \frac{2}{3}t^3 - 3t^2 + 3t$.

c. Substitute $t = 2$ into both expressions. $v(2) = 8 - 12 + 3 = -1$ m s^{-1}; $s(2) = \frac{16}{3} - 12 + 6 = -\frac{2}{3}$ m. The particle is moving back towards the origin.

Answer to Question 5

Method: when the acceleration depends on t, suvat cannot be used. Differentiate to go from displacement to velocity to acceleration, and integrate (adding a constant found from the initial conditions) to go back.

a. Differentiate the velocity. $a = \dfrac{dv}{dt} = 6t - 12$.

b. At rest means $v = 0$; factorise the quadratic. $3t^2 - 12t + 9 = 3(t - 1)(t - 3) = 0$, so $t = 1$ and $t = 3$. So the particle stops at 1 s and at 3 s.

c. Substitute each time into a. At $t = 1$: $a = -6$ m s^{-2}; at $t = 3$: $a = 6$ m s^{-2}. The particle slows and stops at $t = 1$, moves backwards for 1 and stops again at $t = 3$ before moving forwards again.

Answer to Question 6

Method: displacement is $\int v\,dt$, but distance travelled must count every part of the motion as positive. Find when $v = 0$, integrate separately on each interval where v keeps the same sign, and add the sizes of the results.

a. $(t - 2)(t - 4) = 0$, so $t = 2$ and $t = 4$.

b. $s = \int_0^4 (t^2 - 6t + 8)\,dt = \left[\frac{t^3}{3} - 3t^2 + 8t\right]_0^4 = \frac{64}{3} - 48 + 32 = \frac{16}{3}$ m.

c. For $0 \le t \le 2$ the velocity is positive: $\left[\frac{t^3}{3} - 3t^2 + 8t\right]_0^2 = \frac{8}{3} - 12 + 16 = \frac{20}{3}$ m. For $2 \le t \le 4$ it is negative: displacement $\frac{16}{3} - \frac{20}{3} = -\frac{4}{3}$ m. Total distance $= \frac{20}{3} + \frac{4}{3} = 8$ m.

Answer: the displacement is $\frac{16}{3}$ m but the total distance is 8 m, because the particle comes back $\frac{4}{3}$ m after $t = 2$.

Answer to Question 7

a. $u_x = 18\cos 22^\circ = 16.69$, $u_y = 18\sin 22^\circ = 6.743$. At $x = 9.15$: $t = \dfrac{9.15}{16.69} = 0.548$ s. $y = 6.743(0.548) - 4.9(0.548)^2 = 3.697 - 1.473 = 2.22$ m, which is greater than 2 m, so the ball clears the wall.

b. At $x = 20$: $t = 1.198$ s and $y = 6.743 \times 1.198 - 4.9 \times 1.198^2 = 8.08 - 7.04 = 1.04$ m.

c. 1.04, so it goes under the crossbar (and into the goal).

d. Air resistance would reduce the height and the range, so the ball might not clear the wall. Spin (swerve) would change the path sideways and could curve the ball into the goal.

Answer to Question 8

Method: treat the horizontal and vertical motions separately. Horizontally the velocity $u\cos\theta$ is constant; vertically the acceleration is $g = 9.8$ m s^{-2} downwards, so use the suvat equations with initial velocity $u\sin\theta$. The time t links the two motions. The diver leaves horizontally, so the initial vertical velocity is zero.

a. $25 = 4.9t^2$, so $t = \sqrt{5.102} = 2.26$ s.

b. Horizontal distance $= 2 \times 2.26 = 4.52$ m, which is more than 3 m, so the rocks are cleared.

c. $v_y = 9.8 \times 2.26 = 22.1$ m s^{-1}. Speed $= \sqrt{2^2 + 22.1^2} = 22.2$ m s^{-1} at $\tan^{-1}\dfrac{22.1}{2} = 84.8^\circ$ below the horizontal.

Answer: the diver is in the air for about 2.26 s, lands 4.52 m from the cliff (clear of the rocks) and enters the water at about 22.2 m s^{-1}, almost vertically.

Answer to Question 9

Method: treat the horizontal and vertical motions separately. Horizontally the velocity $u\cos\theta$ is constant; vertically the acceleration is $g = 9.8$ m s^{-2} downwards, so use the suvat equations with initial velocity $u\sin\theta$. The time t links the two motions.

a. $u_y = 8\sin 60^\circ = 6.93$. $H = \dfrac{6.93^2}{2 \times 9.8} = 2.45$ m.

b. $T = \dfrac{2 \times 6.93}{9.8} = 1.41$ s and $R = 8\cos 60^\circ \times 1.41 = 5.66$ m.

c. $R \propto u^2$, so doubling R needs u^2 to double: u multiplies by $\sqrt{2} = 1.41$, so $u = 11.3$ m s^{-1}.

Answer: the jet rises about 2.45 m, lands about 5.66 m away, and the speed must be multiplied by $\sqrt{2}$ to double the range.

Answer to Question 10

Method: treat the horizontal and vertical motions separately. Horizontally the velocity $u\cos\theta$ is constant; vertically the acceleration is $g = 9.8$ m s^{-2} downwards, so use the suvat equations with initial velocity $u\sin\theta$. The time t links the two motions. Here the launch point is 1 m above the ground, so the ball lands when $y = 0$ with $y = 1 + u_y t - 4.9t^2$.

a. $u_y = 15\sin 35^\circ = 8.604$. Rise above the launch point $= \dfrac{8.604^2}{19.6} = 3.78$ m, so greatest height $= 4.78$ m.

b. Set $y = 0$: $1 + 8.604t - 4.9t^2 = 0$, so $t = \dfrac{8.604 + \sqrt{74.03 + 19.6}}{9.8} = \dfrac{8.604 + 9.677}{9.8} = 1.87$ s (rejecting the negative root).

c. $x = 15\cos 35^\circ \times 1.865 = 12.29 \times 1.865 = 22.9$ m.

Answer: greatest height about 4.78 m, time of flight about 1.87 s, and the ball lands about 22.9 m from the batter.

Answer to Question 11

Method: when the acceleration depends on t, suvat cannot be used. Differentiate to go from displacement to velocity to acceleration, and integrate (adding a constant found from the initial conditions) to go back.

a. $v = 0.05t(10 - t)$, which is zero at $t = 0$ and $t = 10$.

b. $a = 0.5 - 0.1t = 0$ at $t = 5$, where $v = 2.5 - 1.25 = 1.25$ m s^{-1}.

c. $s = \int_0^{10} (0.5t - 0.05t^2)\,dt = \left[0.25t^2 - \frac{0.05}{3}t^3\right]_0^{10} = 25 - 16.67 = 8.33$ m.

d. $a(8) = 0.5 - 0.8 = -0.3$ m s^{-2}. The acceleration is negative because the lift is slowing down to stop at the top floor.

Answer to Question 12

Method: displacement is $\int v\,dt$, but distance travelled must count every part of the

motion as positive. Find when $v = 0$, integrate separately on each interval where v keeps the same sign, and add the sizes of the results.

a. $(t-2)(t-6) = 0$, so she is at rest at $t = 2$ and $t = 6$. She turns at $t = 2$ (the velocity changes sign there).
b. $\int_0^6 (t^2 - 8t + 12)\,dt = \left[\frac{t^3}{3} - 4t^2 + 12t\right]_0^6 = 72 - 144 + 72 = 0$.
c. For $0 \le t \le 2$: $\frac{8}{3} - 16 + 24 = \frac{32}{3}$ units towards the gate. Then the same distance back. Total $= \frac{64}{3} = 21.3$ units.

Answer to Question 13

Method: when the acceleration depends on t, suvat cannot be used. Differentiate to go from displacement to velocity to acceleration, and integrate (adding a constant found from the initial conditions) to go back.

a. Integrate the acceleration; the train starts from rest, so the constant is zero. $v = \int (2 - 0.1t)\,dt = 2t - 0.05t^2$, using $v = 0$ at $t = 0$. So $v = 2t - 0.05t^2$.
b. Solve $a = 0$ to find the time, then substitute into v. $a = 0$ at $t = 20$: $v = 40 - 20 = 20$ m s^{-1}. So the train reaches 20 m s^{-1} after 20 s, when it stops accelerating.
c. Integrate the velocity from 0 to 20; the velocity stays positive, so this is also the distance. $s = \int_0^{20} (2t - 0.05t^2)\,dt = \left[t^2 - \frac{0.05}{3}t^3\right]_0^{20} = 400 - 133.3 = 266.7$ m. So the train covers about 267 m.

Answer to Question 14

Method: when the acceleration depends on t, suvat cannot be used. Differentiate to go from displacement to velocity to acceleration, and integrate (adding a constant found from the initial conditions) to go back. With vectors, differentiate each component separately.

a. $\mathbf{v} = 2\mathbf{i} + (2t - 4)\mathbf{j}$, $\mathbf{a} = 2\mathbf{j}$ (constant). The acceleration is constant and vertical.
b. Substitute $t = 3$ and find the magnitude of the velocity. $\mathbf{v}(3) = 2\mathbf{i} + 2\mathbf{j}$, speed $= \sqrt{8} = 2.83$ m s^{-1}. So the speed is $2\sqrt{2} \approx 2.83$ m s^{-1}.
c. The $\mathbf{i}$ component never changes, so the speed is least when the $\mathbf{j}$ component is zero. Speed$^2 = 4 + (2t - 4)^2$, smallest when $2t - 4 = 0$, i.e. $t = 2$ s. The least speed is $\sqrt{4} = 2$ m s^{-1}. So the drone is slowest after 2 s, at 2 m s^{-1}.

Answer to Question 15

a. $x = u\cos\theta\, t$ so $t = \dfrac{x}{u\cos\theta}$. Substituting in $y = u\sin\theta\, t - \frac{1}{2}gt^2$ gives

$y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$.

b. Relative to the release point the hoop is at $x = 4.6$, $y = 0.95$.
$0.95 = 4.6\tan 50^\circ - \dfrac{9.8\times 21.16}{2u^2\cos^2 50^\circ}$. So $\dfrac{207.37}{0.8264u^2} = 5.482 - 0.95 = 4.532$, giving $u^2 = \dfrac{207.37}{0.8264\times 4.532} = 55.37$ and $u = 7.44$ m s^{-1}.

c. $t = \dfrac{4.6}{7.44\cos 50^\circ} = 0.962$ s.

d. The ball is higher at every position, so it would pass above the centre of the hoop; the player would need to reduce the speed slightly.

Answer to Question 16

a. $R = \dfrac{625\sin 60^\circ}{9.8} = 55.2$ m.

b. $\sin 2\theta$ takes the same value for θ and $90^\circ - \theta$ because $\sin(180^\circ - 2\theta) = \sin 2\theta$. The maximum of $\sin 2\theta$ is 1, at $2\theta = 90^\circ$, i.e. $\theta = 45^\circ$.

c. $\sin 2\theta = \dfrac{50\times 9.8}{625} = 0.784$, so $2\theta = 51.6^\circ$ or 128.4°, giving $\theta = 25.8^\circ$ or 64.2°.

d. The real range is 52 m, about 6% less than 55.2 m. Air resistance slows the ball, so the model overestimates the range.

Answer to Question 17

a. $t(t-3)^2 = t(t^2 - 6t + 9) = t^3 - 6t^2 + 9t$. The velocity is zero at $t = 0$ and $t = 3$ only, and positive otherwise, so the particle never moves backwards.

b. $s = \int (t^3 - 6t^2 + 9t)\,dt = \frac{t^4}{4} - 2t^3 + \frac{9}{2}t^2$ (with $s = 0$ at $t = 0$).

c. $a = 3t^2 - 12t + 9 = 3(t-1)(t-3)$, which is zero at $t = 1$ in the interval.
$v(1) = 1 - 6 + 9 = 4$ m s^{-1}. Check ends: $v(0) = v(3) = 0$. Greatest speed $= 4$ m s^{-1}.

d. $a(4) = 48 - 48 + 9 = 9$ m s^{-2}; $s(4) = 64 - 128 + 72 = 8$ m.

Answer to Question 18

Method: treat the horizontal and vertical motions separately. Horizontally the velocity $u\cos\theta$ is constant; vertically the acceleration is $g = 9.8$ m s^{-2} downwards, so use the suvat equations with initial velocity $u\sin\theta$. The time t links the two motions.

a. $u_y = 18\sin 40^\circ = 11.57$. Rise above the cliff top $= \dfrac{11.57^2}{19.6} = 6.83$ m, so the greatest height above the sea is 36.8 m.

b. $-30 = 11.57t - 4.9t^2$, so $4.9t^2 - 11.57t - 30 = 0$ and

$t = \dfrac{11.57 + \sqrt{133.9 + 588}}{9.8} = \dfrac{11.57 + 26.87}{9.8} = 3.92$ s.

c. $x = 18\cos 40^\circ \times 3.92 = 13.79 \times 3.922 = 54.1$ m.

d. $v_y = 11.57 - 9.8 \times 3.922 = -26.9$ m s^{-1}, $v_x = 13.79$. Speed $= \sqrt{13.79^2 + 26.87^2} = 30.2$ m s^{-1}, at $\tan^{-1}\dfrac{26.87}{13.79} = 62.8^\circ$ below the horizontal.

Answer to Question 19

a. The coefficient of x is $\tan\theta = 1$, so $\theta = 45^\circ$.

b. $\dfrac{g}{2u^2\cos^2 45^\circ} = 0.05$, so $u^2 = \dfrac{9.8}{0.1 \times 0.5} = 196$ and $u = 14$ m s^{-1}.

c. $y = 0$ when $x(1 - 0.05x) = 0$: $x = 20$ m. The maximum is at $x = 10$ (by symmetry): $y = 10 - 5 = 5$ m.

d. At $x = 15$: $y = 15 - 11.25 = 3.75$ m, which is not 1 m. The jet passes 2.75 m above the flower, so it will not hit it.

Answer to Question 20

Method: treat the horizontal and vertical motions separately. Horizontally the velocity $u\cos\theta$ is constant; vertically the acceleration is $g = 9.8$ m s^{-2} downwards, so use the suvat equations with initial velocity $u\sin\theta$. The time t links the two motions. Use the standard results $R = \dfrac{u^2\sin 2\theta}{g}$ and $H = \dfrac{u^2\sin^2\theta}{2g}$.

a. $R = \dfrac{u^2\sin 2\theta}{g} = \dfrac{2u^2\sin\theta\cos\theta}{g}$ and $H = \dfrac{u^2\sin^2\theta}{2g}$. So $\dfrac{H}{R} = \dfrac{u^2\sin^2\theta}{2g} \times \dfrac{g}{2u^2\sin\theta\cos\theta} = \dfrac{\sin\theta}{4\cos\theta} = \dfrac{\tan\theta}{4}$. Hence $H = \dfrac{R\tan\theta}{4}$.

b. Compute R and H directly with $u = 20$ and $\theta = 30^\circ$: $R = \dfrac{400\sin 60^\circ}{9.8}$ and $H = \dfrac{400 \times 0.25}{19.6}$. $R = 35.3$ m, $H = 5.10$ m and $\dfrac{35.3\tan 30^\circ}{4} = 5.10$ m. They agree. So the formula is confirmed numerically.

Answer to Question 21

Method: when the acceleration depends on t, suvat cannot be used. Differentiate to go from displacement to velocity to acceleration, and integrate (adding a constant found from the initial conditions) to go back. Here the acceleration depends on v, so we separate the variables.

a. $\dfrac{dv}{dt} = -0.5v$, so $\displaystyle\int \frac{dv}{v} = \int -0.5\,dt$, $\ln v = -0.5t + C$. At $t = 0$, $v = 20$, so $C = \ln 20$

and $v = 20e^{-0.5t}$.

b. $10 = 20e^{-0.5t}$, so $e^{-0.5t} = \frac{1}{2}$ and $t = 2\ln 2 = 1.39$ s.

c. $s = \int_0^T 20e^{-0.5t}\,dt = \left[-40e^{-0.5t}\right]_0^T = 40(1 - e^{-0.5T})$ for every T, and tends to 40 as $T \to \infty$.

So the resistance makes the speed halve every 1.39 s, and the particle stops getting further away after about 40 m.