# Lesson - Parametric equations

Year: Year 13

Most curves you have met are given by one equation linking $x$ and $y$. A **parametric** description gives $x$ and $y$ separately in terms of a third variable, often $t$, which is especially useful for circles and for motion where $t$ is time.

**Starter activity**

Imagine a bee flying around a flower garden. At each second a camera records its east-west position and its north-south position. Neither coordinate alone describes the path, but both depend on time. Sketch a possible path for a bee that flies in a perfect circle at constant speed. What would the two coordinates look like as functions of time?

## What are parametric equations?

**Definition: parametric equations**

The equations $x = f(t)$, $y = g(t)$ define a curve; $t$ is the **parameter**. For $x = t + 1$ and $y = t^2$: $t = 2$ gives $(3, 4)$, $t = 0$ gives $(1, 0)$ and $t = -1$ gives $(0, 1)$. As $t$ changes the point traces out the curve.

## Converting to Cartesian form

**Method: eliminating the parameter**

1. Substitution: make $t$ the subject of the simpler equation and substitute into the other.
2. Trigonometric: use $\sin^2 t + \cos^2 t = 1$, $1 + \tan^2 t = \sec^2 t$ or a double angle formula.
3. Test the result with one value of $t$.

**Worked example 1**

**Find the Cartesian equation of (a) $x = 2t - 1$, $y = t^2 + t$; (b) $x = 3t$, $y = \dfrac{2}{t}$.**

(a) $t = \dfrac{x + 1}{2}$, so $y = \dfrac{(x + 1)^2}{4} + \dfrac{x + 1}{2} = \dfrac{x^2 + 4x + 3}{4}$. (b) Multiplying gives $xy = 6$.

**Worked example 2**

**Find the Cartesian equation of (a) $x = 2\cos t$, $y = 3\sin t$; (b) $x = 1 + 2\cos t$,**

$y = -3 + 2\sin t$; **(c)** $x = \sin t$, $y = \cos 2t$.

(a) $\frac{x^2}{4} + \frac{y^2}{9} = 1$, an ellipse. (b) $(x-1)^2 + (y+3)^2 = 4$, a circle with centre $(1, -3)$ and radius 2. (c) $\cos 2t = 1 - 2\sin^2 t$ gives $y = 1 - 2x^2$. Also $x = \sec t$, $y = \tan t$ gives $x^2 - y^2 = 1$, a hyperbola.

## Restrictions and the domain

**Watch out**

The Cartesian equation may describe more of the curve than the parametric equations do. Always find the range of $x$ and $y$ and state any restriction.

**Worked example 3**

**Describe the part of the curve traced by (a) $x = \sin t$, $y = \cos 2t$; (b) $x = e^t$, $y = e^{2t} + 1$; (c) $x = \frac{1}{1+t}$, $y = \frac{t}{1+t}$.**

(a) $-1 \le x \le 1$: only the part of $y = 1 - 2x^2$ between $x = -1$ and $x = 1$. (b) $y = x^2 + 1$, but $e^t > 0$, so only $x > 0$. (c) $x + y = 1$, but $x$ is never 0, so the point $(0, 1)$ is missing.

## Intersections

**Method: intersections**

1. Substitute $x(t)$ and $y(t)$ into the equation of the line or curve.
2. Solve for $t$.
3. Find the coordinates from $t$. For the $x$-axis set $y = 0$; for the $y$-axis set $x = 0$.

**Worked example 4**

**Find where $x = t^2$, $y = 2t$ meets $y = x - 3$.**

$2t = t^2 - 3$, so $t^2 - 2t - 3 = 0$, $(t-3)(t+1) = 0$ and $t = 3$ or $t = -1$. The points are $(9, 6)$ and $(1, -2)$.

**Note: self-intersection**

If several values of $t$ give the same point, the curve crosses itself there. For $x = t^2 - 1$, $y = t^3 - t$, the values $t = 1$ and $t = -1$ both give $(0, 0)$, so the curve has a loop through the origin.

## Modelling with parametric equations

### Worked example 5

**A ball is thrown from height 2 m: $x = 12t$, $y = 2 + 10t - 4.9t^2$. Find the greatest height and where it lands.**

The vertical velocity is zero when $10 - 9.8t = 0$, $t = 1.02$ s, and $y = 2 + \dfrac{100}{19.6} = 7.10$ m. It lands when $y = 0$: $4.9t^2 - 10t - 2 = 0$, so $t = \dfrac{10 + \sqrt{139.2}}{9.8} = 2.22$ s and $x = 26.7$ m. The model ignores air resistance, treats the ball as a particle and assumes constant gravity.

### Worked example 6

**Find the minimum distance from the origin to the curve $x = t + 1$, $y = t - 1$.**

The distance is $\sqrt{(t+1)^2 + (t-1)^2} = \sqrt{2t^2 + 2}$, with minimum $\sqrt{2}$ at $t = 0$.

## Sketching parametric curves

### Method: sketching a parametric curve

1. If you cannot easily eliminate $t$, make a table of values and plot the points in order.
2. Mark the direction in which $t$ increases.
3. Find the axis crossings and the behaviour for large positive and negative $t$.

### Note: a cusp

The curve $x = t^2$, $y = t^3$ has a sharp point (a cusp) at the origin and Cartesian equation $y^2 = x^3$. A point is $(x(t), y(t))$, not $(t, \cdot)$.

### Watch out

Do not forget to restrict the Cartesian equation when $t$ has limits. Square carefully: $\left(\dfrac{x}{2}\right)^2 = \dfrac{x^2}{4}$. If the question asks for a point, do not give only $t$. State circle centres and radii, and give units in modelling questions.

### Worked example 7: Converting with a hyperbola

**Find the Cartesian equation of $x = \sec t$, $y = \tan t$ and state the part of the curve traced for $0 \le t$.**

Use $1 + \tan^2 t = \sec^2 t$: $1 + y^2 = x^2$, so $x^2 - y^2 = 1$, a hyperbola. For $0 \leq t$: $x = \sec t \geq 1$ and $y = \tan t \geq 0$, so only the upper branch on the right ($x \geq 1$, $y \geq 0$) is traced.

### Worked example 8: Parametric equations for a given circle

**Write parametric equations for the circle $(x-2)^2 + (y+1)^2 = 9$.**

The centre is $(2, -1)$ and the radius is 3. Use $x = 2 + 3\cos t$ and $y = -1 + 3\sin t$. Check: $(3\cos t)^2 + (3\sin t)^2 = 9$. For $0 \leq t$ the point makes one anticlockwise circuit starting at $(5, -1)$. Starting elsewhere, or going clockwise, is possible by changing the parameter, so a parametric description is not unique.

### Worked example 9: Finding where a curve meets an axis

**Find where $x = t^2 - 3t$, $y = t + 2$ meets the $x$-axis and the $y$-axis.**

$x$-axis: $y = 0$ gives $t = -2$ and $x = 4 + 6 = 10$: the point $(10, 0)$. $y$-axis: $x = 0$ gives $t(t-3) = 0$, so $t = 0$ or $t = 3$: points $(0, 2)$ and $(0, 5)$. Always substitute back to find both coordinates; the value of $t$ alone is not the answer.

### Worked example 10: Area under a parametric curve

**Find the area between the curve $x = t^2$, $y = t + 1$ for $0 \leq t \leq 2$, the $x$-axis and the lines $x = 0$ and $x = 4$.**

$\int y\,dx = \int_0^2 (t+1)(2t)\,dt = \int_0^2 (2t^2 + 2t)\,dt = \left[\frac{2t^3}{3} + t^2\right]_0^2 = \frac{16}{3} + 4 = \frac{28}{3}$. The limits are values of $t$ ($t = 0$ gives $x = 0$, $t = 2$ gives $x = 4$), so we convert the integral to a $t$-integral using $dx = \frac{dx}{dt}dt$.

### Worked example 11: Gradient from parametric equations

**Find the gradient of the curve $x = t^2 + 1$, $y = t^3 - t$ at the point where $t = 2$.**

$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 1$. So $\frac{dy}{dx} = \frac{3t^2 - 1}{2t}$. At $t = 2$: $\frac{11}{4} = 2.75$. The point is $(5, 6)$, so the tangent is $y - 6 = 2.75(x - 5)$.

### Worked example 12: A projectile modelled parametrically

**A stone is thrown from the top of a cliff: $x = 8t$, $y = 20 + 6t - 4.9t^2$. Find when it hits the sea ($y = 0$) and how far from the base of the cliff it lands.**

$4.9t^2 - 6t - 20 = 0$ gives $t = \frac{6 + \sqrt{36 + 392}}{9.8} = \frac{6 + 20.69}{9.8} = 2.72$ s. Then

$x = 8 \times 2.72 = 21.8$ m. The negative root is rejected because the stone is thrown at $t = 0$.

### Worked example 13: A curve with a fractional parameter

**Find the Cartesian equation of $x = \dfrac{t}{t-1}$, $y = \dfrac{1}{t+2}$ and state any values that are never reached.**

From the second equation, $t + 2 = \dfrac{1}{y}$, so $t = \dfrac{1}{y} - 2 = \dfrac{1-2y}{y}$. Then $t - 1 = \dfrac{1-3y}{y}$ and $x = \dfrac{(1-2y)/y}{(1-3y)/y} = \dfrac{1-2y}{1-3y}$. Rearranging: $x - 3xy = 1 - 2y$, so $y(2-3x) = 1 - x$ and $y = \dfrac{1-x}{2-3x}$. The equation $\dfrac{t}{t-1} = 1$ has no solution, so $x = 1$ is never reached, and $\dfrac{1}{t+2}$ is never zero, so $y = 0$ is never reached. Also $t = 1$ and $t = -2$ are not allowed. The Cartesian equation is therefore valid only for $x \neq 1$ and $y \neq 0$.

### Worked example 14: Direction of travel and speed

**A particle moves on the path $x = 4t$, $y = 3t^2$. Find its speed at $t = 1$ and describe its direction of travel.**

The velocity components are $\dfrac{dx}{dt} = 4$ and $\dfrac{dy}{dt} = 6t$. At $t = 1$ they are 4 and 6, so the speed is $\sqrt{16 + 36} = \sqrt{52} = 7.21$ m/s. The direction makes angle $\tan^{-1}\dfrac{6}{4} = 56.3^\circ$ with the positive $x$-axis. The path is a parabola $y = \dfrac{3x^2}{16}$ and the particle moves to the right as $t$ increases.

### Worked example 15: Sketching from a table

**Sketch the curve $x = t^2 - 2t$, $y = t + 1$ for $-1 \le t \le 3$ by plotting points, then find its Cartesian equation.**

Table: $t = -1$: $(3, 0)$; $t = 0$: $(0, 1)$; $t = 1$: $(-1, 2)$; $t = 2$: $(0, 3)$; $t = 3$: $(3, 4)$. Plotting these gives a parabola opening to the right with vertex at $(-1, 2)$, traced from bottom to top as $t$ increases. Cartesian equation: $t = y - 1$, so $x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$. Check with $t = 3$: $y = 4$, $x = 16 - 16 + 3 = 3$. ✓

### Worked example 16: Comparing two moving objects

**Two drones have positions $\mathbf{r}_1$: $x = 2t$, $y = 10 - t$ and $\mathbf{r}_2$: $x = 14 - t$, $y = 3t - 2$. Do the paths cross, and do the drones collide?**

Paths: eliminate $t$. For drone 1: $t = x/2$, so $y = 10 - x/2$. For drone 2: $t = 14 - x$, so

$y = 3(14 - x) - 2 = 40 - 3x$. Equate: $10 - \frac{x}{2} = 40 - 3x$, so $\frac{5x}{2} = 30$ and $x = 12$, $y = 4$. The paths cross at $(12, 4)$. Drone 1 is there when $2t = 12$, $t = 6$. Drone 2 is there when $14 - t = 12$, $t = 2$. The times differ, so the drones do **not** collide, even though their paths cross.

### Note: parameter versus time

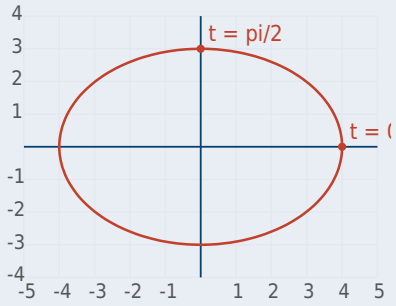
In a modelling question the parameter is usually time, and the speed of the particle is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$. In a pure maths question the parameter may be an angle or have no meaning at all. Two different parametrisations can give the same curve, for example $x = t$, $y = t^2$ and $x = 2s$, $y = 4s^2$, so always say which parametrisation you are using.

### Watch out: common parametric mistakes

- Giving only the value of $t$ when the question asks for the coordinates of a point.
- Forgetting to restrict the Cartesian equation, for example writing $y = 1 - 2x^2$ without noting $-1 \leq x \leq 1$.
- Squaring incorrectly in trigonometric elimination: $\left(\frac{x-2}{3}\right)^2$ has denominator 9.
- Mixing up the parameter and the coordinates when substituting into a line or curve.
- Using degrees on the calculator when $t$ is in radians (or the other way round).

### Note: the circle and the ellipse

The picture shows the ellipse $x = 4\cos t$, $y = 3\sin t$. The parameter $t$ increases as the point moves anticlockwise.




### Method: from parametric to Cartesian in four steps

1. Look at the pair of equations: if both are polynomials or fractions in $t$, make $t$ the subject of the simpler one and substitute.
2. If they contain $\sin t$ and $\cos t$, rearrange to get $\sin t$ and $\cos t$ alone and use $\sin^2 t + \cos^2 t = 1$; for double angles use $\cos 2t = 1 - 2\sin^2 t$ or $\sin 2t = 2\sin t \cos t$

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3. Find the range of $x$ and $y$ from the parametric equations and state any restriction on the Cartesian curve.
4. Test the result with one value of $t$.

**Did you know?**

The cycloid, the curve traced by a point on a rolling wheel, was studied in the 17th century by Galileo, Pascal and Huygens. It has the surprising property that a bead sliding down an upside-down cycloid reaches the bottom in the shortest possible time, whatever the starting point. Parametric equations make cycloids easy to describe, whereas their Cartesian equation is awkward.

**Otto's tip**

When you eliminate the parameter, test your final equation with one value of $t$, for example $t = 1$, to catch algebra slips.