Answers - Parametric equations

Year: Year 13

Answer to Question 1

a. $t = -2$: $(-3, 1)$; $t = -1$: $(-1, -2)$; $t = 0$: $(1, -3)$; $t = 1$: $(3, -2)$; $t = 2$: $(5, 1)$.
b. $2t + 1 = 7$ gives $t = 3$. Check: $y = 9 - 3 = 6$. Yes, the point is $(7, 6)$ when $t = 3$.
c. $x = 0$ needs $t = -0.5$, which gives $y = 0.25 - 3 = -2.75 \neq -3$. So $(0, -3)$ is not on the curve.

Answer to Question 2

Method: to find a Cartesian equation, eliminate the parameter. If one equation is linear in t, make t the subject and substitute. If the equations involve $\sin t$ and $\cos t$, isolate them and use $\sin^2 t + \cos^2 t = 1$ or a double angle formula.

a. The x equation is linear, so make t the subject and substitute into y. $t = \dfrac{x + 2}{3}$, so $y = \dfrac{(x + 2)^2}{9} + 1 = \dfrac{x^2 + 4x + 13}{9}$. The curve is a parabola.
b. Make t the subject of $x = 2t$ and substitute. $t = \dfrac{x}{2}$, so $y = \dfrac{8}{x/2} = \dfrac{16}{x}$, i.e. $xy = 16$. The curve is a rectangular hyperbola; $t = 0$ is not allowed, so $x \neq 0$.

Answer to Question 3

Method: to find a Cartesian equation, eliminate the parameter. If one equation is linear in t, make t the subject and substitute. If the equations involve $\sin t$ and $\cos t$, isolate them and use $\sin^2 t + \cos^2 t = 1$ or a double angle formula.

a. Isolate $\cos t$ and $\sin t$. $\cos t = \dfrac{x}{4}$, $\sin t = \dfrac{y}{3}$. Using $\cos^2 t + \sin^2 t = 1$: $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$, an ellipse. It has semi-axes 4 along the x-axis and 3 along the y-axis.
b. Isolate $\cos t$ and $\sin t$ after moving the constants. $\cos t = \dfrac{x - 2}{3}$, $\sin t = \dfrac{y + 1}{3}$, so $(x - 2)^2 + (y + 1)^2 = 9$: a circle with centre $(2, -1)$ and radius 3. The constants in the parametric equations give the centre.

Answer to Question 4

a. $\cos 2t = 1 - 2\sin^2 t$, so $y = 1 - 2x^2$.
b. $-1 \leq \sin t \leq 1$ so $-1 \leq x \leq 1$. Then x^2 lies between 0 and 1, so y lies between -1

and 1: $-1 \leq y \leq 1$. The curve is only the part of the parabola with $|x| \leq 1$.

Answer to Question 5

Method: to find where a parametric curve meets a line or an axis, substitute the expressions for x and y into the equation of the line, solve for t, then substitute each value of t back to get the coordinates.

a. Replace y by $2t$ and x by $t^2 + 1$ in $y = x - 4$. Substitute: $2t = t^2 + 1 - 4$, so $t^2 - 2t - 3 = 0$. This is the required equation.
b. Factorise the quadratic. $(t - 3)(t + 1) = 0$, so $t = 3$ or $t = -1$. Points: $t = 3$: $(10, 6)$; $t = -1$: $(2, -2)$. Check: $6 = 10 - 4$ and $-2 = 2 - 4$, so both points lie on the line.

Answer to Question 6

a. $y = 0$: $t(t^2 - 4) = 0$, so $t = 0$ or $t = \pm 2$. $t = 0$: $(-4, 0)$; $t = \pm 2$: $(0, 0)$.
b. $x = 0$: $t = \pm 2$, giving $y = 0$ both times: the point $(0, 0)$.
c. The origin is reached for $t = 2$ and $t = -2$: the curve crosses itself at the origin, so it has a loop.

Answer to Question 7

a. Vertical velocity $= 12 - 9.8t = 0$ when $t = 1.2245$ s.
$y = 1 + 12(1.2245) - 4.9(1.2245)^2 = 8.35$ m.
b. $y = 0$: $4.9t^2 - 12t - 1 = 0$, $t = \dfrac{12 + \sqrt{144 + 19.6}}{9.8} = 2.530$ s, so
$x = 15 \times 2.530 = 37.9$ m.
c. At $x = 9$, $t = 0.6$ and $y = 1 + 7.2 - 1.764 = 6.44$ m, which is above 2.2 m: the ball clears the wall.
d. The ball is a particle and there is no air resistance (or spin); gravity is constant at 9.8 m s^{-2}.

Answer to Question 8

Method: in a parametric model, t is usually time. Answer each question by deciding first whether you need a value of t (from a condition on x or y) or a position (by substituting t).

a. Isolate the sine and cosine, square and add.
$x^2 + (y - 12)^2 = 100(\sin^2 + \cos^2) = 100$: a circle with centre $(0, 12)$ and radius 10 m. The centre is 12 m above the foot of the wheel.
b. The height y is smallest when the cosine is 1 and largest when it is -1. Lowest $12 - 10 = 2$ m; highest $12 + 10 = 22$ m. So the car moves between 2 m and 22 m

above the ground.

c. Substitute $t = 5$: the angle is $\frac{5\pi}{20} = \frac{\pi}{4}$. $y = 12 - 10\cos(\pi/4) = 12 - 7.07 = 4.93$ m. So after 5 seconds the car is about 4.93 m above the ground.

d. Solve the inequality for the angle, then convert to time. $12 - 10\cos\theta > 17$ means $\cos\theta$, so $\theta = \frac{\pi t}{20}$ between $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, i.e. t between $\frac{40}{3}$ and $\frac{80}{3}$, a time of $\frac{40}{3} = 13.3$ s. So the car is above 17 m for about 13.3 seconds of each 40-second turn, which is one third of the time.

Answer to Question 9

Method: to find a Cartesian equation, eliminate the parameter. If one equation is linear in t, make t the subject and substitute. If the equations involve $\sin t$ and $\cos t$, isolate them and use $\sin^2 t + \cos^2 t = 1$ or a double angle formula.

a. Use $\cos t = \frac{x}{40}$, $\sin t = \frac{y}{25}$ and $\cos^2 t + \sin^2 t = 1$. $\frac{x^2}{1600} + \frac{y^2}{625} = 1$. This is an ellipse with semi-axes 40 m and 25 m.

b. Use Pythagoras: $d^2 = x^2 + y^2$. $d^2 = 1600\cos^2 t + 625\sin^2 t = 625 + 975\cos^2 t$ (using $\sin^2 t = 1 - \cos^2 t$). This is the required result.

c. Find the extreme values of d^2, then take square roots. $\cos^2 t$ ranges from 0 to 1, so d^2 from 625 to 1600: least distance 25 m, greatest 40 m. These are the semi-axes of the ellipse.

d. Substitute $t = \frac{\pi}{3}$ (that is $60°$). $x = 40\cos 60° = 20$ m, $y = 25\sin 60° = 21.65$ m: $(20, 21.65)$. So the drone is 20 m east and about 21.65 m north of the tower.

Answer to Question 10

Method: in a parametric model, t is usually time. Answer each question by deciding first whether you need a value of t (from a condition on x or y) or a position (by substituting t).

a. Make t the subject of the linear equation and substitute. $t = \frac{x}{5}$ so $y = 12x - \frac{4.9x^2}{25} = 12x - 0.196x^2$. The path is a parabola.

b. The rocket stops rising when $\frac{dy}{dt} = 0$. $60 - 9.8t = 0$ gives $t = 6.12$ s and $y = \frac{3600}{19.6} = 183.7$ m. So the rocket reaches about 184 m after about 6.12 s.

c. Set $y = 0$ and ignore the solution $t = 0$, which is the launch. $y = 0$: $t = \frac{60}{4.9} = 12.24$ s, so $x = 5 \times 12.24 = 61.2$ m. So it would land about 61.2 m away.

d. Substitute the time of the highest point into x. At $t = 6.12$ s, $x = 30.6$ m. This is half of the range, by the symmetry of the parabola.

Answer to Question 11

a. $t = 0$: $(0, 0)$. $t = \pi/2$: $x = 0.35(1.5708 - 1) = 0.20$, $y = 0.35$. $t = \pi$: $x = 0.35\pi = 1.10$, $y = 0.70$. $t = 3\pi/2$: $x = 0.35(4.7124 + 1) = 2.00$, $y = 0.35$. $t = 2\pi$: $x = 2.20$, $y = 0$.
b. Greatest height 0.70 m (the diameter of the wheel). One turn moves the bicycle $0.35 \times 2\pi = 2.20$ m.
c. At the lowest point the reflector touches the road, and a rolling wheel has no slipping, so that point of the wheel has zero velocity relative to the ground.

Answer to Question 12

Method: in a parametric model, t is usually time. Answer each question by deciding first whether you need a value of t (from a condition on x or y) or a position (by substituting t).

a. $t = \dfrac{x}{20}$ gives $y = 30 - 5\dfrac{x^2}{400} = 30 - \dfrac{x^2}{80}$.
b. $y = 0$: $t = \sqrt{6} = 2.449$ s, $x = 20\sqrt{6} = 49.0$ m.
c. $y = 30 - \dfrac{900}{80} = 18.75$ m.
d. A sloping landing area falling away would be reached **later** and further, because the ground is lower below the path.

Answer to Question 13

Method: to find where a parametric curve meets a line or an axis, substitute the expressions for x and y into the equation of the line, solve for t, then substitute each value of t back to get the coordinates.

a. On the line $y = x$, so set $3\sin t = 4\cos t$ and divide by $\cos t$. Substitute: $3\sin t = 4\cos t$, so $\tan t = \dfrac{4}{3}$. This is the required condition.
b. Tangent is positive in the first and third quadrants, so there are two values of t in one turn. $t = 0.9273$ (or $0.9273 + \pi$). $\cos t = 0.6$, $\sin t = 0.8$, so $x = 2.4$, $y = 2.4$: $(2.4, 2.4)$. The other point is $(-2.4, -2.4)$. The two points are symmetric about the origin.
c. Use Pythagoras on the differences of the coordinates. The distance is $\sqrt{4.8^2 + 4.8^2} = 4.8\sqrt{2} = 6.79$ thousand km. So the two points are about 6790 km apart.

Answer to Question 14

Method: in a parametric model, t is usually time. Answer each question by deciding first whether you need a value of t (from a condition on x or y) or a position (by substituting t).

a. Use $d^2 = x^2 + y^2$ and expand both brackets.
$d^2 = (3t-6)^2 + (4t-5)^2 = 9t^2 - 36t + 36 + 16t^2 - 40t + 25 = 25t^2 - 76t + 61$.
This is the required expression.

b. Minimising d^2 is the same as minimising d, so differentiate d^2 with respect to t.
$\frac{d(d^2)}{dt} = 50t - 76 = 0$, so $t = 1.52$ hours. $d^2 = 25(1.52)^2 - 76(1.52) + 61 = 3.24$, so $d = 1.8$ km. So the boat is closest after 1.52 hours (about 1 hour 31 minutes), at a distance of 1.8 km.

c. Compare the minimum distance with the range of the light. 1.8, so yes. So the sailors can see the light at their closest approach, for a short time.

Answer to Question 15

a. $y = x^2 + 1$. Since $e^t > 0$, $x > 0$, and then $y > 1$. Only the part with $x > 0$ is traced.

b. $x + y = \frac{1+t}{1+t} = 1$, so $y = 1 - x$. Since $x = \frac{1}{1+t}$ is never zero, the point $(0, 1)$ is missing.

c. $y = x^2 - 2$. $x = t^2$ ranges over $0 \le x \le 4$ for $-1 \le t \le 2$ (with $x = 0$ at $t = 0$, $x = 1$ at $t = \pm 1$). So only $0 \le x \le 4$ is traced, $-2 \le y \le 14$.

Answer to Question 16

a. $t = 1$: $(0, 0)$; $t = -1$: $(0, 0)$. The curve crosses itself at the origin.

b. $y = 0$ when $t = 0, \pm 1$: points $(-1, 0)$ and $(0, 0)$.

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c. Area .

d. Replacing t by $-t$ changes y to $-y$ but leaves x unchanged, so the curve is symmetrical about the x-axis.

Answer to Question 17

a. $y^2 = t^6 = (t^2)^3 = x^3$.
b. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{2t} = \dfrac{3t}{2}$. At $t = 2$ the gradient is 3.
c. The point is $(4, 8)$, so $y - 8 = 3(x - 4)$, i.e. $y = 3x - 4$.
d. Substitute into $y^2 = x^3$: $(3x - 4)^2 = x^3$, i.e. $x^3 - 9x^2 + 24x - 16 = 0$, which factorises as $(x - 4)^2(x - 1) = 0$. The repeated root $x = 4$ is the tangent point; $x = 1$ gives $y = 3 - 4 = -1$, i.e. $(1, -1)$.

Answer to Question 18

a. $x = 2 + 3t$ matches 2, 5, 8, 11, 14. y: $t = 1$: $1 + 4 - 0.5 = 4.5$; $t = 2$: $1 + 8 - 2 = 7$; $t = 3$: $1 + 12 - 4.5 = 8.5$; $t = 4$: $1 + 16 - 8 = 9$. All match.
b. $t = \dfrac{x - 2}{3}$, so $y = 1 + \dfrac{4(x - 2)}{3} - \dfrac{(x - 2)^2}{18}$.
c. $t = 8$: $x = 26$, $y = 1 + 32 - 32 = 1$. The drone would be back at its starting height after 8 s, having climbed to 9 m at $t = 4$ (maximum), which is plausible for a parabolic flight but not for a drone that can hover or change direction, so the model should only be trusted within the range of the data.

Answer to Question 19

a. $5.625t = 4.5$ gives $t = 0.8$ s.
b. $y = 2 + 5.25(0.8) - 4.9(0.64) = 2 + 4.2 - 3.136 = 3.064$ m.
c. The ball is at 3.064 m, only 0.014 m above the hoop centre, so it passes through the hoop (the ring is 0.45 m wide). The shot is successful.
d. $5.25 - 9.8t = 0$ gives $t = 0.536$ s, $y = 2 + \dfrac{5.25^2}{19.6} = 3.41$ m.
e. The ball is treated as a particle with no air resistance or spin; the ring is treated as a point.

Answer to Question 20

Method: to find a Cartesian equation, eliminate the parameter. If one equation is linear in t, make t the subject and substitute. If the equations involve $\sin t$ and $\cos t$, isolate them and use $\sin^2 t + \cos^2 t = 1$ or a double angle formula.

a. $x^2 + y^2 = \dfrac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} = \dfrac{(1+t^2)^2}{(1+t^2)^2} = 1$.
b. $t = 0$: $(1, 0)$; $t = 1$: $(0, 1)$; $t = -1$: $(0, -1)$.
c. $x = -1$ needs $1 - t^2 = -1 - t^2$, i.e. $1 = -1$, which is impossible. As $t \to \pm\infty$ the point approaches $(-1, 0)$ but never reaches it.

So this parametrisation describes the whole unit circle except the single point $(-1, 0)$.

Answer to Question 21

a. $y = 2\sin t \cos t$, so $y^2 = 4\sin^2 t \cos^2 t = 4x^2(1 - x^2)$.
b. $\sin t = 0$ and $\sin 2t = 0$ for $t = 0, \pi, 2\pi$. At $t = 0$ and $t = \pi$ the curve passes through the origin with different directions, so it crosses itself: a figure of eight.
c. $y = \sin 2t$ is greatest, 1, when $t = \dfrac{\pi}{4}$; then $x = \sin\dfrac{\pi}{4} = 0.707$.