Test answers - Numerical methods

Year: Year 13 · Duration: 40 minutes · /20

Answer to Question 1 (4 marks)

Method (change of sign): if f is continuous on $[a, b]$ and $f(a)$, $f(b)$ have opposite signs, then $f(x) = 0$ has a root between a and b. Always state both the change of sign and the continuity. Mark scheme as shown in brackets.

a. $f(1) = -4 < 0$ and $f(2) = 2 > 0$ (1); f is continuous so there is a root in $[1, 2]$ (1).
b. $f(1.7) = -0.787 < 0$ and $f(1.8) = 0.032 > 0$ (1); continuous, change of sign, root in $[1.7, 1.8]$ (1).

Answer: the root of $x^3 - x - 4 = 0$ lies between 1.7 and 1.8, and it is closer to 1.8 because $|f(1.8)|$ is much smaller than $|f(1.7)|$.

Answer to Question 2 (5 marks)

Mark scheme as shown in brackets.

a. rearrange $f(x) = 0$ as $x = g(x)$, then use $x_{n+1} = g(x_n)$, keeping full calculator accuracy and rounding only when writing the values. The iteration converges near the root α when $|g'(\alpha)| < 1$. Here $x_1 = \sqrt[3]{5.5}$. $x_1 = 1.7652$, $x_2 = 1.7931$, $x_3 = 1.7960$ (2). The values increase towards the root in a staircase pattern.
b. $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Each step follows the tangent at x_n down to the x-axis. Use the ANS key on the calculator and stop when two values agree to the required accuracy. Here $f(1.8) = 0.032$ and $f'(1.8) = 8.72$. $f\prime(x) = 3x^2 - 1$ (1). $x_1 = 1.8 - \dfrac{0.032}{8.72} = 1.79633$ (1). $x_2 = 1.79632$ (1). The two values agree to 4 decimal places, so $\alpha = 1.7963$ to 4 decimal places.

Answer to Question 3 (4 marks)

Method (trapezium rule): $\displaystyle\int_a^b y\,dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})]$ with strip width $h = \dfrac{b-a}{n}$. The estimate is too large when the curve is convex and too small when it is concave. Mark scheme as shown in brackets.

$h = 0.25$ and ordinates $1, 1.0308, 1.1180, 1.2500, 1.4142$ (1).
$\dfrac{0.25}{2}[1 + 1.4142 + 2(1.0308 + 1.1180 + 1.2500)]$ (1) $= 0.125 \times 9.2118 = 1.1515$ (1). The curve is convex, so the estimate is too large (1).

Answer: the integral is approximately 1.1515, which is a slight overestimate (the exact

value is about 1.1478).

Answer to Question 4 (3 marks)

Method (Newton-Raphson): $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Each step follows the tangent at x_n down to the x-axis. Use the ANS key on the calculator and stop when two values agree to the required accuracy. Mark scheme as shown in brackets.

$f\prime(x) = 3x^2 - 2$ (1). $x_1 = 0 - \dfrac{2}{-2} = 1$, $x_2 = 1 - \dfrac{1}{1} = 0$, $x_3 = 1$ (1). The iteration cycles between 0 and 1 and never converges to the root near -1.77 (1).

This shows that Newton-Raphson can fail when the starting value is badly chosen; starting at $x_0 = -2$ instead would converge quickly to the root -1.769.

Answer to Question 5 (4 marks)

Method (fixed-point iteration): rearrange $f(x) = 0$ as $x = g(x)$, then use $x_{n+1} = g(x_n)$, keeping full calculator accuracy and rounding only when writing the values. The iteration converges near the root α when $|g'(\alpha)| < 1$. Mark scheme as shown in brackets.

a. Write the cube root as a power $\frac{1}{3}$ and use the chain rule. $g(x) = (x+4)^{1/3}$ so $g\prime(x) = \dfrac{1}{3}(x+4)^{-2/3}$ (2).

b. Substitute $\alpha = 1.796$, so $x + 4 = 5.796$. $g\prime(1.796) = \dfrac{1}{3(5.796)^{2/3}} = 0.103$ (1). Since $0 < g\prime(\alpha) < 1$ the iteration converges, approaching α from one side in a staircase (1). The error is multiplied by about 0.1 at each step, so the convergence is fast.