Worksheet - Numerical methods

Year: Year 13 · 20

Show your working.

Question 1

Question 1 – Locating a root *(Section A · Skills practice)*

$f(x) = x^3 + 2x - 7$.

a. Show that $f(x) = 0$ has a root in the interval $[1, 2]$. [2 marks]
b. By evaluating $f(1.5)$ and $f(1.6)$, show that the root lies between 1.5 and 1.6. [2 marks]
c. Explain why the change of sign of $g(x) = \dfrac{1}{x - 1}$ between $x = 0$ and $x = 2$ does not mean there is a root. [2 marks]

Question 2

Question 2 – Using an iteration formula *(Section A · Skills practice)*

The equation $x^3 + 2x - 7 = 0$ can be rearranged as $x = \sqrt[3]{7 - 2x}$.

a. Use $x_{n+1} = \sqrt[3]{7 - 2x_n}$ with $x_0 = 1.5$ to find $x_1, \ldots, x_5$ to 4 decimal places. [3 marks]
b. Describe the pattern of the values. [1 mark]
c. Write down the root to 2 decimal places. [1 mark]

Question 3

Question 3 – Newton-Raphson *(Section A · Skills practice)*

$f(x) = x^3 + 2x - 7$ has a root near 1.5.

a. Find $f\prime(x)$. [1 mark]
b. Use the Newton-Raphson formula with $x_0 = 1.5$ to find x_1 and x_2 to 5 decimal places. [4 marks]
c. Hence state the root correct to 4 decimal places. [1 mark]

Question 4

Question 4 – The trapezium rule *(Section A · Skills practice)*

Use the trapezium rule with 4 strips to estimate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

a. Complete a table of ordinates to 4 decimal places. [2 marks]
b. Calculate the estimate to 4 decimal places. [3 marks]
c. The exact value is $\ln 2 = 0.6931$. State whether the estimate is too large or too small and explain why. [2 marks]

Question 5

Question 5 – Convergence of an iteration *(Section A · Skills practice)*

The equation $x = \cos x$ has one root $\alpha = 0.7391$ (4 d.p.). Consider $x_{n+1} = \cos x_n$ with $x_0 = 1$.

a. Calculate $x_1, \ldots, x_5$ to 4 decimal places (radians). [3 marks]
b. Find $g'(x)$ where $g(x) = \cos x$ and evaluate $g'(\alpha)$. [2 marks]
c. Explain why the iteration converges. [1 mark]

Question 6

Question 6 – Two roots *(Section A · Skills practice)*

The equation $e^x = 3x$ has two positive roots. Let $f(x) = e^x - 3x$.

a. Show that one root lies in $[0.6, 0.7]$. [2 marks]
b. Show that the other root lies in $[1.5, 1.6]$. [2 marks]
c. Explain how a sketch of $y = e^x$ and $y = 3x$ shows there are exactly two roots. [2 marks]

Question 7

Question 7 – The pizza box *(Section B · Real-life practice)*

A square sheet of card of side 30 cm has squares of side x cm cut from each corner. The sides are folded up to make an open box (a pizza tray).

a. Show that the volume of the box is $V = x(30 - 2x)^2$. [2 marks]
b. The volume must be 1000 cm^3. Show that a suitable value of x lies between 1 and 1.5. [3 marks]
c. Use Newton-Raphson on $f(x) = x(30 - 2x)^2 - 1000$ with $x_0 = 1.2$ to find x to 3 decimal places. [4 marks]
d. Give one reason why $x = 10$ is also a solution but not practical. [1 mark]

Question 8

Question 8 – The cost of a loan *(Section B · Real-life practice)*

A customer borrows 1000 pounds and repays it in 5 equal annual instalments of 250 pounds. The annual interest rate r (as a decimal) satisfies

$$f(r) = 250\,\frac{1-(1+r)^{-5}}{r} - 1000 = 0.$$

a. Show that r lies between 0.07 and 0.08. [3 marks]
b. By evaluating $f(0.079)$ and $f(0.0795)$, find the rate to 3 decimal places. [4 marks]
c. Express the rate as a percentage and comment on whether it is a sensible rate for a loan. [1 mark]

Question 9

Question 9 – A river cross-section *(Section B · Real-life practice)*

The depth of a river is measured every 2 m across its width.

Distance from bank (m)	0	2	4	6	8	10	12
Depth (m)	0	1.2	2.5	3.1	2.8	1.6	0

a. Use the trapezium rule with 6 strips to estimate the cross-sectional area of the river. [4 marks]
b. The water flows at an average speed of 0.8 m/s. Estimate the volume of water passing per second. [2 marks]
c. State whether the trapezium rule would give an over- or under-estimate if the bed were smooth and convex downward, and why. [1 mark]

Question 10

Question 10 – Fuel used on a journey *(Section B · Real-life practice)*

The fuel flow rate of a delivery van is recorded every 10 minutes during a 60-minute journey.

Time (min)	0	10	20	30	40	50	60
Rate (litres per minute)	0.05	0.08	0.11	0.12	0.10	0.07	0.04

a. Use the trapezium rule to estimate the total fuel used. [4 marks]
b. The van tank held 20 litres. Estimate the percentage of the tank used. [2 marks]
c. Explain what the area under the rate-time graph represents. [1 mark]

Question 11

Question 11 – Water in a spherical tank *(Section B · Real-life practice)*

A spherical tank of radius 1.5 m is partly filled with water to a depth h m. The volume of water is $V = \dfrac{\pi h^2(4.5 - h)}{3}$ cubic metres. The tank must contain exactly 3 m^3.

a. Show that $h^2(4.5 - h) = \dfrac{9}{\pi}$. [2 marks]

b. Show that h lies between 0.89 and 0.9. [3 marks]
c. Use Newton-Raphson on $f(h) = h^2(4.5 - h) - \frac{9}{\pi}$, with $h_0 = 1$, to find h to 3 decimal places. [4 marks]

Question 12

Question 12 – A bacteria culture *(Section B · Real-life practice)*

A biologist models the time x (hours) at which a culture reaches a target size by $x + \ln x = 3$.

a. Show that the equation has a root in $[2, 3]$. [2 marks]
b. Show that the equation can be rearranged as $x = 3 - \ln x$. [1 mark]
c. Use $x_{n+1} = 3 - \ln x_n$ with $x_0 = 2$ to find $x_1, \ldots, x_6$ to 4 decimal places. [3 marks]
d. Write the root to 2 decimal places and explain why the iteration converges. [3 marks]

Question 13

Question 13 – When Newton-Raphson goes wrong *(Section B · Real-life practice)*

A playground designer models the profile of a slide by $f(x) = x^3 - 2x + 2$ and wants the point where it meets the ground, $f(x) = 0$.

a. Show that $f(x) = 0$ has a root in $[-2, -1.5]$. [3 marks]
b. Use Newton-Raphson with $x_0 = 0$ and write down x_1, x_2, x_3, x_4. [4 marks]
c. Explain what has gone wrong and suggest a better starting value. [2 marks]

Question 14

Question 14 – A complete write-up *(Section C · Problem solving)*

$f(x) = x^3 - 4x - 3$.

a. Show that $f(x) = 0$ has a root α in the interval $[2.3, 2.31]$. [2 marks]
b. The equation can be written as $x = \sqrt[3]{4x + 3}$. Use $x_{n+1} = \sqrt[3]{4x_n + 3}$ with $x_0 = 2$ to find x_1, x_2, x_3, x_4 to 4 decimal places. [3 marks]
c. Use Newton-Raphson with $x_0 = 2.3$ to find α correct to 4 decimal places, showing x_1 and x_2. [4 marks]
d. By considering the sign of $f(2.3025)$ and $f(2.3035)$ prove that $\alpha = 2.303$ to 3 decimal places. [3 marks]

Question 15

Question 15 – Choosing a rearrangement *(Section C · Problem solving)*

The equation $x^3 - x - 1 = 0$ has a root $\alpha \approx 1.3247$. It can be rearranged in two ways:

(A) $x = x^3 - 1$ and (B) $x = \sqrt[3]{x+1}$.

a. Use each formula with $x_0 = 1.5$ to find x_1, x_2, x_3 (A) and $x_1, \ldots, x_5$ (B). [4 marks]
b. Find the gradient function of each rearrangement and evaluate it at α. [4 marks]
c. Hence explain which rearrangement is suitable and why the other fails. [2 marks]
d. Describe the pattern of values for (B) and sketch the staircase diagram in words. [2 marks]

Question 16

Question 16 – How accurate is the trapezium rule? *(Section C · Problem solving)*

The integral $I = \int_0^1 e^{-x^2}\,dx$ cannot be found exactly by elementary methods. Its value to 4 decimal places is 0.7468.

x	0	0.25	0.5	0.75	1
e^{-x^2}	1	0.9394	0.7788	0.5698	0.3679

a. Use the trapezium rule with 2 strips to estimate I. [3 marks]
b. Use the trapezium rule with 4 strips to estimate I. [3 marks]
c. Calculate the error of each estimate. Comment on how the error changes when the number of strips doubles. [3 marks]
d. State whether you expect the estimate to be too large or too small, giving a reason. [1 mark]

Question 17

Question 17 – Newton-Raphson and starting values *(Section C · Problem solving)*

Let $f(x) = e^x - 3x$, with roots near 0.62 and 1.51.

a. Write down the Newton-Raphson iteration for f. [2 marks]
b. Starting with $x_0 = 0.5$, find x_1, x_2, x_3 to 5 decimal places. [3 marks]
c. Starting with $x_0 = 2$, find x_1, x_2, x_3, x_4 to 5 decimal places. [3 marks]
d. Starting with $x_0 = 1$, find x_1 and x_2 and explain the difficulty. [4 marks]

Question 18

Question 18 – The circular segment *(Section C · Problem solving)*

A circular window of radius 5 dm has a segment cut off by a chord, as shown by the shaded region. The angle at the centre is θ radians and the segment has area 12 dm^2.

a. Show that $\theta - \sin\theta = 0.96$. [3 marks]
b. Show that θ lies between 1.9 and 2.0. [3 marks]
c. Use Newton-Raphson from $\theta_0 = 1.8$ to find θ to 3 decimal places. [4 marks]

d. Hence find the length of the chord. [2 marks]

Question 19

Question 19 – Challenge: the error in Newton-Raphson for a square root *(Section D · Challenge)*

To find $\sqrt{a}$ Newton-Raphson is applied to $f(x) = x^2 - a$.

a. Show that $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. [2 marks]

b. Show that $x_{n+1} - \sqrt{a} = \frac{(x_n - \sqrt{a})^2}{2x_n}$. [3 marks]

c. Starting with $x_0 = 2.5$, use the formula to estimate $\sqrt{7}$ to 6 decimal places. Explain why the number of correct digits roughly doubles. [4 marks]

Question 20

Question 20 – Challenge: an exact formula for the trapezium error *(Section D · Challenge)*

The trapezium rule with n strips is applied to $\int_0^1 x^2\,dx = \frac{1}{3}$.

a. Show that the estimate is $T_n = \frac{1}{3} + \frac{1}{6n^2}$. [5 marks]

b. Find T_4 and T_8 and the ratio of their errors. [2 marks]

c. Explain why the trapezium rule always overestimates this integral. [1 mark]