# Answers - Numerical methods

Year: Year 13

## Answer to Question 1

a. $f(1) = 1 + 2 - 7 = -4$ and $f(2) = 8 + 4 - 7 = 5 > 0$. The function is continuous and there is a change of sign, so there is a root in $[1, 2]$.
b. $f(1.5) = 3.375 + 3 - 7 = -0.625$ and $f(1.6) = 4.096 + 3.2 - 7 = 0.296 > 0$. Change of sign and continuity give a root in $[1.5, 1.6]$.
c. $g(0) = -1$ and $g(2) = 1$, but $g$ is not continuous on $[0, 2]$ because it is undefined at $x = 1$. The graph jumps from $-\infty$ to $+\infty$ and never crosses the axis, so there is no root.

## Answer to Question 2

Method (fixed-point iteration): rearrange $f(x) = 0$ as $x = g(x)$, then use $x_{n+1} = g(x_n)$, keeping full calculator accuracy and rounding only when writing the values. The iteration converges near the root $\alpha$ when $|g'(\alpha)|$.

a. $x_1 = 1.5874$, $x_2 = 1.5639$, $x_3 = 1.5703$, $x_4 = 1.5686$, $x_5 = 1.5690$.
b. The values alternate above and below the root and get closer together (a cobweb pattern, the gradient of $g$ is negative).
c. The root is $1.57$ to 2 decimal places.

Answer: the iterates oscillate around and close in on the root, which is 1.57 to 2 decimal places.

## Answer to Question 3

Method (Newton-Raphson): $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Each step follows the tangent at $x_n$ down to the $x$-axis. Use the ANS key on the calculator and stop when two values agree to the required accuracy.

a. Differentiate term by term. $f\prime(x) = 3x^2 + 2$.
b. First $f(1.5) = 3.375 + 3 - 7 = -0.625$ and $f'(1.5) = 8.75$.
$x_1 = 1.5 - \dfrac{-0.625}{3(2.25) + 2} = 1.5 + \dfrac{0.625}{8.75} = 1.57143$.
$x_2 = 1.57143 - \dfrac{f(1.57143)}{f\prime(1.57143)} = 1.56895$. The two values already agree to 2 decimal places, which shows how fast the method converges.
c. A further step gives $x_3 = 1.568946$, which agrees with $x_2$ to 4 decimal places. The root is $1.5689$ (4 d.p.). Check: $f(1.56885)$ and $f(1.56895) > 0$, so the root is 1.5689 to 4 decimal places.

## Answer to Question 4

Method (trapezium rule): $\int_a^b y\,dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})]$ with strip width $h = \frac{b-a}{n}$. The estimate is too large when the curve is convex and too small when it is concave.

a. $h = 0.25$: $x = 1, 1.25, 1.5, 1.75, 2$ gives $y = 1, 0.8, 0.6667, 0.5714, 0.5$.
b. $\frac{0.25}{2}[1 + 0.5 + 2(0.8 + 0.6667 + 0.5714)] = 0.125 \times 5.5762 = 0.6970$.
c. The estimate $0.6970$ is **too large**. The curve $y = \frac{1}{x}$ is convex (bends upwards), so the straight chord tops lie above the curve.

## Answer to Question 5

Method (fixed-point iteration): rearrange $f(x) = 0$ as $x = g(x)$, then use $x_{n+1} = g(x_n)$, keeping full calculator accuracy and rounding only when writing the values. The iteration converges near the root $\alpha$ when $|g'(\alpha)|$.

a. Use radian mode and press $\cos$ repeatedly on the previous answer. $x_1 = 0.5403$, $x_2 = 0.8576$, $x_3 = 0.6543$, $x_4 = 0.7935$, $x_5 = 0.7014$. The values jump above and below 0.7391.
b. Differentiate $\cos x$. $g\prime(x) = -\sin x$, so $g\prime(0.7391) = -0.6736$.
c. Compare the size of the gradient at the root with 1. $|g\prime(\alpha)| = 0.6736$, so the iteration converges, and since $g\prime$ it alternates (cobweb). So each step multiplies the error by about $-0.67$, and the values converge slowly while alternating.

## Answer to Question 6

a. $f(0.6) = 1.8221 - 1.8 = 0.0221 > 0$ and $f(0.7) = 2.0138 - 2.1 = -0.0862$; continuous, so a root lies in $[0.6, 0.7]$.
b. $f(1.5) = 4.4817 - 4.5 = -0.0183$ and $f(1.6) = 4.9530 - 4.8 = 0.1530 > 0$; continuous, so a root lies in $[1.5, 1.6]$.
c. The line $y = 3x$ starts below the curve at the origin ($e^0 = 1 > 0$), crosses it, falls below it again and then the exponential curve overtakes the line permanently. That gives two intersections.

## Answer to Question 7

a. The base is a square of side $30 - 2x$ and the height is $x$, so $V = x(30 - 2x)^2$.
b. $f(1) = 784 - 1000 = -216$ and $f(1.5) = 1.5 \times 729 - 1000 = 93.5 > 0$. Change of sign, continuous, so the root lies in $[1, 1.5]$.

c. $f\prime(x) = (30-2x)^2 - 4x(30-2x)$. $x_1 = 1.2 - \dfrac{f(1.2)}{f\prime(1.2)} = 1.3365$, $x_2 = 1.3397$, $x_3 = 1.33975$. So $x = 1.340$ cm to 3 d.p.

d. $x = 10$ gives a base of side 10 cm and height 10 cm, a deep narrow box, whereas a tray needs low sides: it is a valid solution of the equation but gives a different shape.

## Answer to Question 8

Method (change of sign): if $f$ is continuous on $[a, b]$ and $f(a)$, $f(b)$ have opposite signs, then $f(x) = 0$ has a root between $a$ and $b$. Always state both the change of sign and the continuity.

a. Evaluate $f$ at both ends of the interval.
$f(0.07) = 250 \times 4.1002 - 1000 = 25.05 > 0$ and
$f(0.08) = 250 \times 3.9927 - 1000 = -1.82$. Change of sign and $f$ continuous for $r > 0$. So the rate lies between 7% and 8%.

b. Evaluate $f$ at both values; the sign changes, so the root lies between them.
$f(0.079) = 0.81 > 0$ and $f(0.0795) = -0.51$ so $r$ lies in $[0.079, 0.0795]$, giving $r = 0.079$ to 3 d.p. (more precisely 0.0793). Every value in this interval rounds to 0.079.

c. Multiply the decimal by 100. About 7.9% per year, which is typical for a personal loan.

## Answer to Question 9

a. $h = 2$. Area $\approx \dfrac{2}{2}[0 + 0 + 2(1.2 + 2.5 + 3.1 + 2.8 + 1.6)] = 2 \times 11.2 = 22.4$ m$^2$.

b. Volume per second $= 22.4 \times 0.8 = 17.9$ m$^3$/s.

c. For a bed with the shape of a bowl, the cross-section is concave (bends downwards when depth is plotted upward), so the chords lie below the curve and the rule **underestimates** the area.

## Answer to Question 10

Method (trapezium rule): $\displaystyle\int_a^b y\,dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})]$ with strip width $h = \dfrac{b-a}{n}$. The estimate is too large when the curve is convex and too small when it is concave.

a. The strip width is $h = 10$ minutes; the first and last ordinates are counted once and the others twice.
$\dfrac{10}{2}[0.05 + 0.04 + 2(0.08 + 0.11 + 0.12 + 0.10 + 0.07)] = 5 \times (0.09 + 0.96) = 5.25$

litres. So the van uses about 5.25 litres of fuel.

b. Divide the fuel used by the capacity of the tank. $\frac{5.25}{20} = 26.25\% \approx 26\%$. So the van uses about a quarter of its tank.

c. Think about the units: litres per minute times minutes. The area is the total volume of fuel used, because rate $\times$ time gives volume.

## Answer to Question 11

Method (Newton-Raphson): $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Each step follows the tangent at $x_n$ down to the $x$-axis. Use the ANS key on the calculator and stop when two values agree to the required accuracy.

a. Set the volume equal to 3 and multiply both sides by $\frac{3}{\pi}$. $\frac{\pi h^2(4.5-h)}{3} = 3$ gives $h^2(4.5-h) = \frac{9}{\pi} = 2.8648$. This is the required equation.

b. Use the change of sign method: if $f$ is continuous on $[a, b]$ and $f(a)$, $f(b)$ have opposite signs, then $f(x) = 0$ has a root between $a$ and $b$. Always state both the change of sign and the continuity. $f(0.89) = 0.7921 \times 3.61 - 2.8648 = -0.0053$ and $f(0.9) = 0.81 \times 3.6 - 2.8648 = 0.0512 > 0$. Change of sign, continuous: root in $[0.89, 0.9]$. So the depth is between 0.89 m and 0.9 m.

c. Expand $f(h) = 4.5h^2 - h^3 - \frac{9}{\pi}$ to differentiate it. $f\prime(h) = 9h - 3h^2$. $h_1 = 1 - \frac{0.6352}{6} = 0.8941$, $h_2 = 0.8909$, $h_3 = 0.8909$. So $h = 0.891$ m to 3 d.p. (89.1 cm). The last two values agree, so the depth is about 0.891 m.

## Answer to Question 12

Method (fixed-point iteration): rearrange $f(x) = 0$ as $x = g(x)$, then use $x_{n+1} = g(x_n)$, keeping full calculator accuracy and rounding only when writing the values. The iteration converges near the root $\alpha$ when $|g'(\alpha)|$.

a. $f(x) = x + \ln x - 3$: $f(2) = 2 + 0.6931 - 3 = -0.3069$ and $f(3) = 3 + 1.0986 - 3 = 1.0986 > 0$. Continuous, so there is a root in $[2, 3]$.

b. $x + \ln x = 3 \Rightarrow x = 3 - \ln x$.

c. $x_1 = 2.3069$, $x_2 = 2.1641$, $x_3 = 2.2280$, $x_4 = 2.1989$, $x_5 = 2.2120$, $x_6 = 2.2061$.

d. The root is $2.21$ to 2 d.p. (2.2079). $g\prime(x) = -\frac{1}{x}$ and $|g\prime(2.21)| = 0.45$, so the iteration converges; the negative sign gives the alternating pattern.

## Answer to Question 13

a. $f(-2) = -8 + 4 + 2 = -2$ and $f(-1.5) = -3.375 + 3 + 2 = 1.625 > 0$, continuous:

a root in $[-2, -1.5]$.

b. $f\prime(x) = 3x^2 - 2$. $x_1 = 0 - \dfrac{2}{-2} = 1$; $x_2 = 1 - \dfrac{1}{1} = 0$; $x_3 = 1$; $x_4 = 0$.

c. The iteration cycles between 0 and 1 and never approaches the root, because the tangent at $x = 1$ sends it back to 0. Choose a start closer to the root, such as $x_0 = -1.8$, which converges to $-1.7693$.

## Answer to Question 14

a. $f(2.3) = 12.167 - 9.2 - 3 = -0.033$ and $f(2.31) = 12.326 - 9.24 - 3 = 0.086 > 0$. Continuous, so there is a root in $[2.3, 2.31]$.

b. $x_1 = \sqrt[3]{11} = 2.2240$, $x_2 = \sqrt[3]{11.896} = 2.2828$, $x_3 = 2.2977$, $x_4 = 2.3015$ (approaching 2.3028 from below).

c. $f\prime(x) = 3x^2 - 4$. $x_1 = 2.3 - \dfrac{-0.033}{11.87} = 2.3027801$, $x_2 = 2.3027756$. So $\alpha = 2.3028$ (4 d.p.).

d. $f(2.3025) = -0.0033$ and $f(2.3035) = 0.0086 > 0$. Continuous with a change of sign, so $\alpha$ lies in $[2.3025, 2.3035]$, and every number in this interval rounds to 2.303. So $\alpha = 2.303$ to 3 d.p.

## Answer to Question 15

Method (fixed-point iteration): rearrange $f(x) = 0$ as $x = g(x)$, then use $x_{n+1} = g(x_n)$, keeping full calculator accuracy and rounding only when writing the values. The iteration converges near the root $\alpha$ when $|g'(\alpha)|$.

a. (A): $x_1 = 2.375$, $x_2 = 12.396$, $x_3 = 1904.0$. (B): $x_1 = 1.3572$, $x_2 = 1.3309$, $x_3 = 1.3259$, $x_4 = 1.3249$, $x_5 = 1.3248$.

b. (A): $g\prime(x) = 3x^2$, $g\prime(\alpha) = 3(1.3247)^2 = 5.26$. (B): $g\prime(x) = \dfrac{1}{3(x+1)^{2/3}}$, $g\prime(\alpha) = 0.19$.

c. (B) converges because $|g\prime(\alpha)| = 0.19$. (A) diverges because $|g\prime(\alpha)| = 5.26 > 1$ so the errors grow each step.

d. In (B) the values decrease towards $\alpha$ from above: $0$ gives a staircase. Each step reduces the error by about 81%.

## Answer to Question 16

a. $h = 0.5$: $\dfrac{0.5}{2}[1 + 0.3679 + 2(0.7788)] = 0.25 \times 2.9255 = 0.7314$.

b. $h = 0.25$:
$\dfrac{0.25}{2}[1 + 0.3679 + 2(0.9394 + 0.7788 + 0.5698)] = 0.125 \times 5.9439 = 0.7430$.

c. Errors: $0.7468 - 0.7314 = 0.0154$ and $0.7468 - 0.7430 = 0.0038$. Doubling the number of strips divides the error by about 4, which is typical of the trapezium

rule.

d. Both estimates are **too small**: on most of $[0, 1]$ the curve is concave (bends downwards), so the chords lie below the curve.

## Answer to Question 17

a. $x_{n+1} = x_n - \dfrac{e^{x_n} - 3x_n}{e^{x_n} - 3}$.

b. $x_1 = 0.61006$, $x_2 = 0.61900$, $x_3 = 0.61906$ (root $0.619$).

c. $x_1 = 1.68352$, $x_2 = 1.54348$, $x_3 = 1.51349$, $x_4 = 1.51214$ (root $1.512$).

d. $f\prime(1) = e - 3 = -0.2817$ and $f(1) = e - 3 = -0.2817$, so $x_1 = 1 - \dfrac{-0.2817}{-0.2817} = 0$. Then $f\prime(0) = -2$, $f(0) = 1$, so $x_2 = 0 + \dfrac{1}{2} = 0.5$. The start $x_0 = 1$ is close to the minimum of $f$ (where $f\prime = 0$ at $x = \ln 3 = 1.0986$), so the tangent is almost horizontal and the first step is poor: it throws the iteration to the other root region.

## Answer to Question 18

Method (Newton-Raphson): $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Each step follows the tangent at $x_n$ down to the $x$-axis. Use the ANS key on the calculator and stop when two values agree to the required accuracy.

a. Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta) = 12.5(\theta - \sin\theta) = 12$, so $\theta - \sin\theta = \dfrac{12}{12.5} = 0.96$.

b. $g(\theta) = \theta - \sin\theta - 0.96$: $g(1.9) = 1.9 - 0.9463 - 0.96 = -0.0063$ and $g(2.0) = 2 - 0.9093 - 0.96 = 0.1307 > 0$. Continuous with a change of sign, so $\theta \in [1.9, 2.0]$.

c. $g\prime = 1 - \cos\theta$. $\theta_1 = 1.8 - \dfrac{-0.13385}{1.22720} = 1.9091$, $\theta_2 = 1.90476$, $\theta_3 = 1.90475$. So $\theta = 1.905$ (3 d.p.).

d. Chord $= 2r\sin\dfrac{\theta}{2} = 10\sin(0.95238) = 8.15$ dm.

Answer: $\theta \approx 1.905$ radians and the chord is about 8.15 dm long.

## Answer to Question 19

a. $x_{n+1} = x_n - \dfrac{x_n^2 - a}{2x_n} = \dfrac{2x_n^2 - x_n^2 + a}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$.

b. $x_{n+1} - \sqrt{a} = \dfrac{x_n^2 + a - 2x_n\sqrt{a}}{2x_n} = \dfrac{(x_n - \sqrt{a})^2}{2x_n}$.

c. $x_1 = \frac{1}{2}(2.5 + 2.8) = 2.65$, $x_2 = \frac{1}{2}(2.65 + 2.641509) = 2.645755$, $x_3 = 2.645751$ (6 d.p.), and $\sqrt{7} = 2.645751$. The error is squared (and divided by about $2x_n$) at each step, so if the error is $10^{-k}$ the next error is about $10^{-2k}$: the number of correct digits doubles.

## Answer to Question 20

a. $h = \dfrac{1}{n}$ and $T_n = \dfrac{h}{2}\left[0 + 1 + 2\sum_{k=1}^{n-1}\dfrac{k^2}{n^2}\right]$. Using $\sum_{k=1}^{n-1} k^2 = \dfrac{(n-1)n(2n-1)}{6}$:
$T_n = \dfrac{1}{2n}\left[1 + \dfrac{(n-1)(2n-1)}{3n}\right] = \dfrac{1}{2n}\cdot\dfrac{3n + 2n^2 - 3n + 1}{3n} = \dfrac{2n^2+1}{6n^2} = \dfrac{1}{3} + \dfrac{1}{6n^2}$.

b. $T_4 = \dfrac{1}{3} + \dfrac{1}{96} = 0.34375$ and $T_8 = \dfrac{1}{3} + \dfrac{1}{384} = 0.33594$. The errors are $\dfrac{1}{96}$ and $\dfrac{1}{384}$: the ratio is 4.

c. $y = x^2$ is convex, so each chord lies above the curve and every trapezium has more area than the strip under the curve.