Test answers - Integration techniques and differential equations

Year: Year 13 · Duration: 40 minutes · /20

Answer to Question 1 (4 marks)

Method: for a linear inside function $ax+b$, integrate as usual and divide by a: $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+c$, $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|+c$, $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)+c$. Always check by differentiating. Mark scheme: 1 mark for each integral, 1 mark for each correct use of the limits.

$\int e^{2x}dx = \frac{1}{2}e^{2x}$ (1) and $\int \frac{1}{2x+1}dx = \frac{1}{2}\ln(2x+1)$ (1). Limits: $\frac{1}{2}(e^2-1)$ (1) and $\frac{1}{2}\ln 3$ (1). Total $\frac{e^2-1}{2}+\frac{\ln 3}{2} = 3.74$.

Answer: the exact value is $\frac{e^2-1+\ln 3}{2} \approx 3.74$.

The modulus is not needed in $\ln|2x+1|$ here because $2x+1>0$ on $[0,1]$.

Answer to Question 2 (4 marks)

Method (substitution): choose u, find du in terms of dx, change the limits to values of u, rewrite the whole integral in u, then integrate. With new limits there is no need to go back to x. Mark scheme as shown in brackets.

$du = 2x\,dx$ (1). Limits: $x=0 \Rightarrow u=3$; $x=1 \Rightarrow u=4$ (1). $\int_3^4 u^{1/2}du = \left[\frac{2}{3}u^{3/2}\right]_3^4$ (1) $= \frac{2}{3}(8-3\sqrt{3}) = \frac{16-6\sqrt{3}}{3} = 1.87$ (1).

Answer: the integral equals $\frac{16-6\sqrt{3}}{3} \approx 1.87$.

Answer to Question 3 (4 marks)

Method (integration by parts): $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Choose u as the factor that becomes simpler when differentiated (a power of x), except with $\ln x$, which should always be u. Mark scheme as shown in brackets.

$u=\ln x$, $v=\frac{x^3}{3}$ (1). $\left[\frac{x^3}{3}\ln x\right]_1^e - \int_1^e \frac{x^2}{3}dx$ (1) $= \frac{e^3}{3} - \left[\frac{x^3}{9}\right]_1^e$ (1) $= \frac{2e^3+1}{9}$ (1).

Answer: $\int_1^e x^2 \ln x \, dx = \dfrac{2e^3 + 1}{9} \approx 4.57$.

Check: $\ln e = 1$ and $\ln 1 = 0$, so the boundary term is $\frac{e^3}{3}$.

Answer to Question 4 (4 marks)

Method: write $3x + 5 = A(x + 2) + B(x + 1)$. Putting $x = -1$ gives $A = 2$, and putting $x = -2$ gives $-1 = -B$, so $B = 1$. Then integrate each fraction to a logarithm and combine with the laws of logarithms. Mark scheme as shown in brackets.

$\dfrac{3x + 5}{(x + 1)(x + 2)} = \dfrac{2}{x + 1} + \dfrac{1}{x + 2}$ (2).

$[2\ln(x + 1) + \ln(x + 2)]_2^3 = (2\ln 4 + \ln 5) - (2\ln 3 + \ln 4)$ (1) $= \ln \dfrac{16 \cdot 5}{9 \cdot 4} = \ln \dfrac{20}{9}$ (1).

Answer: the integral equals $\ln \dfrac{20}{9} \approx 0.799$, so $k = \dfrac{20}{9}$.

Check the partial fractions at $x = 0$: $\frac{5}{2} = 2 + \frac{1}{2}$ ✓. Leaving the answer as $2\ln 4 + \ln 5 - 2\ln 3 - \ln 4$ loses the final mark.

Answer to Question 5 (4 marks)

Method (separating the variables): put all the y terms with dy on one side and all the x (or t) terms on the other, integrate both sides, add one constant, then use the initial condition to find it. Mark scheme as shown in brackets.

$\int \dfrac{1}{y^2} dy = \int 2x \, dx$ (1), so $-\dfrac{1}{y} = x^2 + c$ (1). $y = 1$ at $x = 0$ gives $c = -1$, so $y = \dfrac{1}{1 - x^2}$ (1). It is valid for $-1 < x < 1$, since $y \to \infty$ at $x = \pm 1$ (1).

Answer: $y = \dfrac{1}{1 - x^2}$ for $-1 < x < 1$.