Lesson - Integration techniques and differential equations

Year: Year 13

Integration reverses differentiation. At A level you add exponential, reciprocal and trigonometric integrals, learn substitution, integration by parts and partial fractions, and solve separable differential equations.

Starter activity

Draw a velocity-time graph for a car that accelerates steadily from rest. How can you find the distance travelled from the graph? Now suppose the speed is described by a formula such as $v = 3te^{-0.5t}$. Why is it hard to find the area under this curve by counting squares, and what would an exact method need to do?

Integrals you must know

Rule: standard integrals

- $\int e^{kx}\,dx = \dfrac{1}{k}e^{kx} + c$ and $\int \dfrac{1}{x}\,dx = \ln|x| + c$
- $\int \sin kx\,dx = -\dfrac{1}{k}\cos kx + c$ and $\int \cos kx\,dx = \dfrac{1}{k}\sin kx + c$
- $\int \sec^2 x\,dx = \tan x + c$
- Reverse chain rule: $\int \dfrac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

Worked example 1

Evaluate $\int_0^1 e^{2x}\,dx$ and find $\int \dfrac{1}{2x+1}\,dx$, $\int \tan x\,dx$ and $\int \sin^2 x\,dx$.

$\left[\dfrac{1}{2}e^{2x}\right]_0^1 = \dfrac{e^2-1}{2} \approx 3.19$. Then $\frac{1}{2}\ln|2x+1| + c$ and $-\ln|\cos x| + c$. For $\sin^2 x = \dfrac{1-\cos 2x}{2}$: $\dfrac{x}{2} - \dfrac{\sin 2x}{4} + c$.

Watch out

$\int \dfrac{1}{2x+1}\,dx$ needs the factor $\frac{1}{2}$. $\int \sin kx\,dx$ carries a minus sign. Always include $+c$.

Integration as the limit of a sum

Definition: integral as a limit of a sum

$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$: the area under a curve found by adding thin rectangles of width δx and height $f(x)$.

Worked example 2

Evaluate $\lim_{\delta x \to 0} \sum_{x=1}^{4} x^2 \delta x$.

It equals $\int_1^4 x^2 dx = \left[\frac{x^3}{3}\right]_1^4 = 21$.

Integration by substitution

Method: substitution

1. Choose $u = g(x)$ so that the integrand contains $g'(x)$ (or a multiple of it).
2. Find $du = g'(x)\,dx$ and replace everything in terms of u.
3. For a definite integral change the limits to u values, and write them down immediately.
4. Integrate; for an indefinite integral substitute back.

Worked example 3

Find $\int 2x(x^2+1)^3 dx$ and $\int_0^1 x\sqrt{1+x^2}\,dx$.

$u = x^2+1$, $du = 2x\,dx$: $\int u^3 du = \frac{(x^2+1)^4}{4} + c$. For the second $u = 1+x^2$ gives $\frac{1}{2}\int_1^2 u^{1/2} du = \frac{2\sqrt{2}-1}{3} \approx 0.609$.

Worked example 4

Evaluate $\int_0^3 \frac{x}{\sqrt{x+1}} dx$.

Let $u = x+1$, $x = u-1$, $dx = du$. Limits: $x = 0 \Rightarrow u = 1$; $x = 3 \Rightarrow u = 4$. The integral is $\int_1^4 \left(u^{1/2} - u^{-1/2}\right) du = \left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^4 = \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) = \frac{8}{3}$.

Integration by parts

Rule: integration by parts

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Choose u as the function that becomes simpler when differentiated, following LATE: Logarithm, Algebraic, Trigonometric, Exponential.

Worked example 5

Find $\int xe^x dx$ and $\int_0^\pi x\sin x\,dx$.

With $u = x$, $\frac{dv}{dx} = e^x$: $xe^x - \int e^x dx = (x-1)e^x + c$. For the second, $u = x$, $v = -\cos x$: $[-x\cos x]_0^\pi + \int_0^\pi \cos x\,dx = \pi + 0 = \pi$.

Worked example 6

Find $\int \ln x\,dx$ and $\int x^2 e^x dx$.

Take $u = \ln x$, $\frac{dv}{dx} = 1$: $x\ln x - \int 1\,dx = x\ln x - x + c$, so $\int_1^e \ln x\,dx = 1$. Repeating the method gives $\int x^2 e^x dx = e^x(x^2 - 2x + 2) + c$.

Integration using partial fractions

Worked example 7

Evaluate $\int_2^3 \frac{5x+1}{(x-1)(x+2)}dx$.

$\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$, so the integral is $[2\ln|x-1| + 3\ln|x+2|]_2^3 = 2\ln 2 + 3\ln 5 - 3\ln 4 = 3\ln 5 - 4\ln 2 \approx 2.056$. Use the laws of logarithms for a simple exact answer.

Differential equations

Definition: separable differential equation

A differential equation of the form $\frac{dy}{dx} = f(x)g(y)$. A **general solution** contains a constant; a **particular solution** uses a given condition.

Method: separating the variables

1. Separate: $\int \frac{1}{g(y)}dy = \int f(x)dx$.
2. Integrate both sides, with one constant c.
3. Use the initial condition to find c.
4. Make y the subject if asked, and interpret in context.

Worked example 8

Solve $\frac{dy}{dx} = 2xy$ with $y = 3$ when $x = 0$.

$\int \frac{1}{y}dy = \int 2x\,dx$, so $\ln y = x^2 + c$, $y = Ae^{x^2}$, and $A = 3$: $y = 3e^{x^2}$.

Worked example 9

A population satisfies $\frac{dP}{dt} = 0.05P$. When does it double?

$P = P_0 e^{0.05t}$, and $e^{0.05t} = 2$ gives $t = 20\ln 2 \approx 13.9$. (Newton's law of cooling $\frac{d\theta}{dt} = -k(\theta - 20)$ gives $\theta - 20 = Ae^{-kt}$.)

Worked example 10

A tank empties with $\frac{dh}{dt} = -0.2\sqrt{h}$, depth h m, t in minutes, $h = 4$ at $t = 0$. Find h and the time to empty.

$\int h^{-1/2}dh = \int -0.2\,dt$, so $2\sqrt{h} = -0.2t + c$. With $h = 4$ at $t = 0$, $c = 4$. Then $\sqrt{h} = 2 - 0.1t$ and $h = (2 - 0.1t)^2$. The tank is empty after 20 minutes. The formula is only valid for $0 \le t \le 20$: afterwards it would give the depth rising again, which is impossible.

Note: validity of models

Real populations cannot grow forever, and real objects cool towards room temperature, not below it. Comment on this in exam modelling questions.

How to choose a method

Method: choosing an integration method

1. A function multiplied by (a multiple of) its own derivative: substitution or the reverse chain rule.
2. A product of two unrelated functions such as x and $\sin x$: by parts.

3. A fraction with a factorised denominator: partial fractions.
4. Otherwise rewrite with an identity or a division. Always check by differentiating.

Watch out

Do not forget $+c$, or to use the initial condition. Change the limits (or substitute back) with substitution. A bad choice of u in integration by parts makes the integral harder.

Worked example 11: A sum that becomes an integral

Write $\lim_{\delta x \to 0} \sum_{x=0}^{2} (3x^2 + 1)\delta x$ as an integral and evaluate it.

The sum of thin rectangles of height $3x^2 + 1$ and width δx becomes $\int_0^2 (3x^2 + 1)dx = \left[x^3 + x\right]_0^2 = 8 + 2 = 10$. The sigma notation is just a way of saying that we add up the areas of the strips.

Worked example 12: A trigonometric integral using an identity

Find $\int \cos^2 x \, dx$ and evaluate $\int_0^{\pi/2} \cos^2 x \, dx$.

Use $\cos^2 x = \dfrac{1 + \cos 2x}{2}$. Then $\int \cos^2 x \, dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + c$. Between 0 and $\frac{\pi}{2}$: $\dfrac{\pi}{4} + 0 - 0 = \dfrac{\pi}{4}$. Notice that you cannot integrate $\cos^2 x$ directly; rewrite it first.

Worked example 13: Spotting a reverse chain rule

Evaluate $\int_0^{\pi/4} \tan x \sec^2 x \, dx$.

Let $u = \tan x$, so $du = \sec^2 x \, dx$. Limits: $x = 0 \Rightarrow u = 0$; $x = \frac{\pi}{4} \Rightarrow u = 1$. $\int_0^1 u \, du = \frac{1}{2}$. The integrand is a function ($\tan x$) multiplied by its own derivative ($\sec^2 x$), which is the signal for substitution.

Worked example 14: Parts applied to a trigonometric product

Find $\int x \cos 2x \, dx$.

$u = x$, $\dfrac{dv}{dx} = \cos 2x$, $v = \frac{1}{2}\sin 2x$.
$\int x \cos 2x \, dx = \frac{1}{2}x \sin 2x - \frac{1}{2}\int \sin 2x \, dx = \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x + c$. Check by differentiating: $\frac{1}{2}\sin 2x + x \cos 2x - \frac{1}{2}\sin 2x = x \cos 2x$.

Worked example 15: An exact answer using partial fractions and logarithms

Show that $\int_1^2 \dfrac{4}{(x+1)(x+3)} dx = 2 \ln \dfrac{6}{5}$.

$\dfrac{4}{(x+1)(x+3)} = \dfrac{2}{x+1} - \dfrac{2}{x+3}$. Integrating: $\left[2\ln\dfrac{x+1}{x+3}\right]_1^2 = 2\ln\dfrac{3}{5} - 2\ln\dfrac{2}{4} = 2\ln\dfrac{3/5}{1/2} = 2\ln\dfrac{6}{5} = 0.365$. Combining logarithms before substituting the limits makes the algebra shorter.

Worked example 16: Newton law of cooling with a negative temperature drop

A metal bar at $200°$C is placed in a room at $20°$C. Given that $\dfrac{d\theta}{dt} = -0.05(\theta - 20)$, find the time for the bar to cool to $100°$C.

The solution is $\theta = 20 + 180e^{-0.05t}$. For $\theta = 100$: $e^{-0.05t} = \dfrac{80}{180} = \dfrac{4}{9}$, so $t = \dfrac{\ln(9/4)}{0.05} = 16.2$ minutes.

Rule: choosing u in integration by parts

The aim of integration by parts is to turn a hard integral into an easier one. Pick u to be the factor that becomes simpler (or at least no worse) when differentiated, and let $\dfrac{dv}{dx}$ be the factor you can integrate easily. For $\int x^n \ln x\,dx$, take $u = \ln x$ because integrating $\ln x$ directly is harder; for $\int x^n e^{kx}dx$ or $\int x^n \sin kx\,dx$, take $u = x^n$. When the power of x is 2, apply the method twice; each application reduces the power by one. For $\int e^x \sin x\,dx$ the integral reappears after two applications, and you solve for it as an equation.

Worked example 17: Parts with a logarithm and a definite integral

Evaluate $\int_1^4 \dfrac{\ln x}{\sqrt{x}}dx$.

Take $u = \ln x$, $\dfrac{dv}{dx} = x^{-1/2}$, so $v = 2\sqrt{x}$. Then $\int \dfrac{\ln x}{\sqrt{x}}dx = 2\sqrt{x}\ln x - \int 2\sqrt{x}\cdot\dfrac{1}{x}dx = 2\sqrt{x}\ln x - \int 2x^{-1/2}dx = 2\sqrt{x}\ln x - 4\sqrt{x}$. Between 1 and 4: $(4\ln 4 - 8) - (0 - 4) = 4\ln 4 - 4 = 8\ln 2 - 4 = 1.55$. Checking by differentiating the antiderivative confirms the result.

Worked example 18: A definite integral with substitution and exponential

Use $u = e^x$ to evaluate $\int_0^{\ln 3} \dfrac{e^x}{1+e^{2x}}dx$.

With $u = e^x$, $du = e^x dx$. Limits: $x = 0 \Rightarrow u = 1$; $x = \ln 3 \Rightarrow u = 3$. The integral

becomes $\int_1^3 \frac{1}{1+u^2}du = \left[\tan^{-1} u\right]_1^3 = \tan^{-1} 3 - \frac{\pi}{4} = 0.4636$. Recognising $\frac{1}{1+u^2}$ as the derivative of $\tan^{-1} u$ is the key step.

Worked example 19: Mean value of a function

The mean value of $f(x)$ over $a \le x \le b$ is $\frac{1}{b-a}\int_a^b f(x)dx$. Find the mean value of $f(x) = e^{-x}$ over $0 \le x \le 2$.

$\int_0^2 e^{-x}dx = [-e^{-x}]_0^2 = 1 - e^{-2}$. The width is 2, so the mean value is $\frac{1-e^{-2}}{2} = 0.432$.

This is the height of a rectangle on the same base with the same area as the region under the curve.

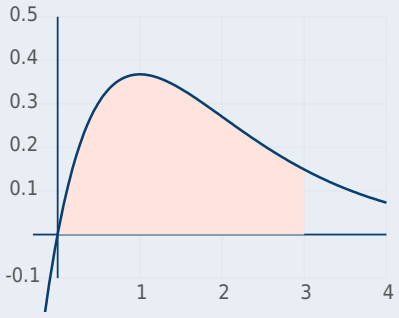
Worked example 20: Two stages: separating the variables with a trigonometric function

Solve $\frac{dy}{dx} = y\cos x$ with $y = 2$ when $x = 0$.

$\int \frac{1}{y}dy = \int \cos x\,dx$, so $\ln y = \sin x + c$. With $y = 2$ at $x = 0$, $c = \ln 2$, so $\ln y - \ln 2 = \sin x$ and $y = 2e^{\sin x}$. The solution oscillates between $2e^{-1} = 0.74$ and $2e = 5.44$.

Note: the area under y = x e^(-x)

Integration by parts gives $\int xe^{-x}dx = -(x+1)e^{-x} + c$. The graph shows the area under the curve from 0 to 3, which equals $1 - 4e^{-3}$.



Forming differential equations from a description

Definition: rate of change models

A sentence such as "the rate at which the height decreases is proportional to the square root of the height" translates directly into a differential equation. "Rate of

change of h" means $\dfrac{dh}{dt}$, "proportional to" introduces a constant k, and "decreases" gives a minus sign, so the model is $\dfrac{dh}{dt} = -k\sqrt{h}$ with $k > 0$. Two pieces of information are then needed: one fixes the constant of integration and the other fixes k. Always state what each letter represents and its units before you start solving.

Worked example 21: a leaking water tank

Water leaks from a tank so that $\dfrac{dh}{dt} = -k\sqrt{h}$, where h m is the depth after t minutes. The depth is 4 m at the start and 1 m after 10 minutes. Find h in terms of t and the time when the tank is empty.

Separate the variables: $\int h^{-1/2}dh = \int -k\,dt$, so $2\sqrt{h} = -kt + c$. At $t = 0$, $h = 4$, so $c = 2 \times 2 = 4$. At $t = 10$, $h = 1$: $2 = -10k + 4$, so $k = 0.2$. Hence $2\sqrt{h} = 4 - 0.2t$, which gives $h = (2 - 0.1t)^2$. The tank is empty when $2 - 0.1t = 0$, that is after $t = 20$ minutes. The model is only valid for $0 \le t \le 20$; after that the formula would wrongly predict the depth rising again.

Watch out: areas below the x-axis

A definite integral measures signed area. Where the curve lies below the x-axis the integral is negative, so $\int_0^{2\pi} \sin x\,dx = 0$ even though the shaded region is clearly not empty. To find a total area, split the interval at each root, integrate each part separately and add the absolute values: the area between $y = \sin x$ and the axis from 0 to 2π is $2 + 2 = 4$.

Method: parts, substitution or partial fractions

1. Is the integrand a function times (a multiple of) its own derivative, or does it have a compound function inside? Try a substitution.
2. Is it a product of two unrelated functions, such as $x\sin x$ or $x\ln x$? Use integration by parts, choosing u by LATE.
3. Is it a fraction with a factorised denominator and a lower-degree numerator? Use partial fractions.
4. Otherwise rewrite using an identity ($\sin^2 x$, $\cos^2 x$) or algebraic division.
5. Check by differentiating the answer, and never forget limits and $+c$.

Watch out: common integration errors

- Forgetting the $\frac{1}{k}$ factor when integrating e^{kx}, $\sin kx$ or $\dfrac{1}{kx + a}$.
- Leaving old limits in x after a substitution; change them to u limits at once.
- Choosing $u = e^x$ or $u = \sin x$ in parts when the polynomial or logarithm should

be u.
- Writing $\ln(a-b)$ instead of $\ln a - \ln b$ and other log law slips in exact answers.
- Not using the initial condition to find the constant in a differential equation.

Did you know?

Integration by parts comes from the product rule: if you integrate $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ and rearrange, you obtain the formula. The integral sign $\int$ is an elongated letter S, introduced by Leibniz in 1675, standing for "summa", the sum. Differential equations of the form $\frac{dy}{dx} = ky$ govern radioactive decay, population growth and the cooling of a cup of tea.

Otto's tip

After any definite integral by substitution, write the new limits next to the u integral immediately, so you never forget them.