Worksheet - Integration techniques and differential equations

Year: Year 13 · 21

Show your working.

Question 1

Question 1 – Standard integrals *(Section A · Skills practice)*

Find each integral, or evaluate it where limits are given.

a. $\int e^{3x}\,dx$ [1 mark]

b. $\int \dfrac{1}{2x+5}\,dx$ [2 marks]

c. $\int \sin 4x\,dx$ [2 marks]

d. $\int_0^{\pi/2} \sin 2x\,dx$ [3 marks]

Question 2

Question 2 – Reverse chain rule *(Section A · Skills practice)*

Use the reverse chain rule $\int \dfrac{f\prime(x)}{f(x)}dx = \ln|f(x)| + c$ to find:

a. $\int \dfrac{2x}{x^2+3}\,dx$ [2 marks]

b. $\int \tan x\,dx$ [3 marks]

c. $\int_0^1 \dfrac{e^x}{e^x+1}\,dx$, giving an exact answer [3 marks]

Question 3

Question 3 – Integration by substitution *(Section A · Skills practice)*

Use a suitable substitution to evaluate each integral.

a. $\int_0^2 x(x^2+1)^3\,dx$ using $u = x^2+1$ [4 marks]

b. $\int_0^1 xe^{x^2}\,dx$ using $u = x^2$ [4 marks]

Question 4

Question 4 – Substitution with a shift *(Section A · Skills practice)*

Use the substitution $u = x+1$ to evaluate $\int_0^3 x\sqrt{x+1}\,dx$.

Question 5

Question 5 – Integration by parts *(Section A · Skills practice)*

Use integration by parts to find or evaluate:

a. $\int xe^{2x}\,dx$ [4 marks]
b. $\int_1^e x\ln x\,dx$ [5 marks]

Question 6

Question 6 – Partial fractions *(Section A · Skills practice)*

Express $\dfrac{7x+1}{(x-1)(x+3)}$ in partial fractions and hence evaluate $\int_2^4 \dfrac{7x+1}{(x-1)(x+3)}\,dx$, giving your answer in the form $a\ln 3 + b\ln\dfrac{7}{5}$.

Question 7

Question 7 – Water flowing into a tank *(Section B · Real-life practice)*

Water flows into a tank at a rate of $R = 12e^{-0.2t}$ litres per minute, where t is the time in minutes after a tap is opened.

a. Find the rate when $t = 0$ and when $t = 10$. [2 marks]
b. Find the volume of water that flows in during the first 10 minutes. [3 marks]
c. Show that the volume flowing in can never exceed 60 litres. [3 marks]

Question 8

Question 8 – Distance travelled by a sprinter *(Section B · Real-life practice)*

The speed of a sprinter during the first 4 seconds of a race is modelled by $v = 3te^{-0.5t}$ m s^{-1}, where t is the time in seconds.

a. Find the time at which the speed is greatest and state this speed. [4 marks]
b. Use integration by parts to find the distance run in the first 4 seconds. [5 marks]
c. Find the average speed over the first 4 seconds. [1 mark]

Question 9

Question 9 – A cooling cup of coffee *(Section B · Real-life practice)*

The temperature θ degrees Celsius of a cup of coffee in a room at 20°C satisfies the law of cooling $\dfrac{d\theta}{dt} = -k(\theta - 20)$, where t is in minutes and $k > 0$. Initially $\theta = 90$ and after 10 minutes $\theta = 55$.

a. Solve the differential equation to show that $\theta = 20 + 70e^{-kt}$. [4 marks]
b. Find the exact value of k. [3 marks]
c. The coffee is drinkable below 30°C. Find when this happens. [3 marks]

Question 10

Question 10 – Deer population *(Section B · Real-life practice)*

A herd of deer in a national park has population P at time t years, with $\dfrac{dP}{dt} = 0.04P$.

Initially there are 500 deer.

a. Solve the differential equation. [4 marks]
b. Find how long it takes the herd to double. [3 marks]
c. Give two reasons why the model may become unrealistic. [2 marks]

Question 11

Question 11 – A draining tank *(Section B · Real-life practice)*

Water drains from a cylindrical tank so that the depth h metres at time t minutes satisfies $\dfrac{dh}{dt} = -k\sqrt{h}$ for a positive constant k. At $t = 0$, $h = 9$, and at $t = 5$, $h = 4$.

a. Solve the differential equation to show that $2\sqrt{h} = 6 - kt$. [4 marks]
b. Find k. [2 marks]
c. Find when the tank is empty. [2 marks]
d. Explain why the formula is not valid after this time. [1 mark]

Question 12

Question 12 – Ticket sales *(Section B · Real-life practice)*

A theatre sells tickets for a new show at a rate of $S(t) = 40te^{-0.2t}$ tickets per day, where t is the number of days since sales opened.

t	0	5	10	15	20
S	0	73.6	54.1	29.9	14.7

a. Find the exact number of tickets sold in the first 10 days. [5 marks]
b. The theatre holds 1200 seats. Show that the show can never sell out. [3 marks]
c. Find the day on which the rate of sales is greatest. [2 marks]

Question 13

Question 13 – Average power *(Section B · Real-life practice)*

The power in an electrical circuit is $P = P_0 \sin^2 t$, where t is the time in seconds and P_0

is a constant in watts.

a. Use the identity $\sin^2 t = \dfrac{1 - \cos 2t}{2}$ to find $\int_0^{\pi} \sin^2 t \, dt$. [4 marks]

b. Hence find the mean value of $\sin^2 t$ over one half-cycle $0 \le t \le \pi$ and the mean power. [3 marks]

Question 14

Question 14 – Pollution in a lake *(Section B · Real-life practice)*

A factory discharges a pollutant into a lake. The mass m kg of pollutant in the lake after t days satisfies $\dfrac{dm}{dt} = 0.5 - 0.02m$, with $m = 0$ when $t = 0$.

a. Show that $m = 25\left(1 - e^{-0.02t}\right)$. [5 marks]

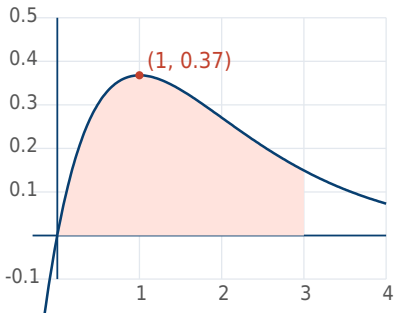
b. Find the long-term mass of pollutant. [1 mark]

c. Find when the mass reaches 20 kg. [3 marks]

Question 15

Question 15 – Area under a curve *(Section C · Problem solving)*

The diagram shows the curve $y = xe^{-x}$. The region R is bounded by the curve, the x-axis and the line $x = 3$.



a. Find the coordinates of the maximum point of the curve. [4 marks]

b. Use integration by parts to show that the area of R is $1 - 4e^{-3}$. [5 marks]

c. Evaluate the area to 3 significant figures. [1 mark]

d. Use the trapezium rule with 3 strips to estimate the area and comment on the accuracy. [3 marks]

Question 16

Question 16 – Integral as the limit of a sum *(Section C · Problem solving)*

A rectangular tank of base 4 m^2 is filled with water to a depth that varies as the tank is lowered into a pool. The area of a horizontal cross-section at height x metres is $A(x) = x^2$ square metres for $0 \le x \le 3$.

a. Explain, by dividing the height into thin slices of thickness δx, why the volume of the solid is $\lim_{\delta x \to 0} \sum_{x=0}^{3} x^2 \delta x$. [3 marks]
b. Evaluate this limit. [2 marks]
c. The tank is filled by a pump at 0.5 m^3 per minute. Find the time to fill the volume. [2 marks]
d. Use a sum of 3 slices of thickness 1 m (using the left-hand ends) to estimate the volume, and explain why it is an underestimate. [3 marks]

Question 17

Question 17 – A growing population with partial fractions *(Section C · Problem solving)*

A population P (in thousands) of fish in a lake satisfies $\dfrac{dP}{dt} = 0.02P(5 - P)$, where t is in years. Initially $P = 1$.

a. Show that $\dfrac{1}{P(5-P)} = \dfrac{1}{5}\left(\dfrac{1}{P} + \dfrac{1}{5-P}\right)$. [2 marks]
b. Solve the differential equation to show that $P = \dfrac{5e^{0.1t}}{4 + e^{0.1t}}$. [7 marks]
c. Find when the population reaches 4000. [3 marks]
d. State the long-term population and explain the shape of the model. [2 marks]

Question 18

Question 18 – An arch and a substitution *(Section C · Problem solving)*

A footbridge arch has height $y = \dfrac{x}{\sqrt{4 - x^2}}$ metres above a horizontal line at a horizontal distance x metres, for $0 \le x \le 1.5$ in a design drawing. Part of the drawing is used for a stained glass panel.

a. Use the substitution $u = 4 - x^2$ to find $\displaystyle\int \frac{x}{\sqrt{4 - x^2}}\,dx$. [4 marks]
b. Find the area under the curve between $x = 0$ and $x = 1.5$. [3 marks]
c. Show that the area between $x = 0$ and $x = 2$ would be exactly 2 m^2, and explain why the integral up to 2 needs care. [3 marks]

Question 19

Question 19 – Total charge *(Section C · Problem solving)*

The current in a circuit after a switch is closed is $I = t^2 e^{-t}$ amperes, where t is in seconds. The charge that has flowed is $Q = \int_0^T I\,dt$ coulombs.

a. Use integration by parts twice to show that $\int t^2 e^{-t} dt = -e^{-t}\left(t^2 + 2t + 2\right) + c$. [6

marks]
b. Find the charge in the first 3 seconds. [2 marks]
c. Find the total charge as $T \to \infty$ and comment. [2 marks]

Question 20

Question 20 – Challenge: a cyclic integral *(Section D · Challenge)*

Let $J = \int_0^{\pi/2} e^x \sin x \, dx$.

a. Use integration by parts twice to show that $2J = e^{\pi/2} + 1$. [6 marks]
b. Find the value of J to 3 significant figures. [1 mark]

Question 21

Question 21 – Challenge: a symmetric trick *(Section D · Challenge)*

Let $K = \int_0^{\pi/2} \dfrac{\sin x}{\sin x + \cos x} dx$ and $L = \int_0^{\pi/2} \dfrac{\cos x}{\sin x + \cos x} dx$.

a. Show that $K + L = \dfrac{\pi}{2}$. [2 marks]
b. Use the substitution $u = \dfrac{\pi}{2} - x$ to show that $K = L$. [4 marks]
c. Hence find the exact value of K. [1 mark]