Answers - Integration techniques and differential equations

Year: Year 13

Answer to Question 1

Method: for a linear inside function $ax+b$, integrate as usual and divide by a: $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+c$, $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|+c$, $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)+c$. Always check by differentiating.

a. Here $a=3$. $\frac{1}{3}e^{3x}+c$. Check: differentiating gives e^{3x}.

b. Here $a=2$ and the integral of $\frac{1}{u}$ is $\ln|u|$. $\frac{1}{2}\ln|2x+5|+c$ (the factor $\frac{1}{2}$ comes from the inside function $2x+5$). So the answer is $\frac{1}{2}\ln|2x+5|+c$.

c. The integral of $\sin$ is $-\cos$, and here $a=4$. $-\frac{1}{4}\cos 4x+c$. Check: $\frac{d}{dx}\left(-\frac{1}{4}\cos 4x\right) = \sin 4x$.

d. Integrate, then substitute the upper limit and subtract the value at the lower limit. $\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = -\frac{1}{2}\cos\pi + \frac{1}{2}\cos 0 = \frac{1}{2}+\frac{1}{2} = 1$. So the area under one arch of $\sin 2x$ is exactly 1.

Answer to Question 2

a. The derivative of x^2+3 is exactly $2x$. The numerator is the derivative of the denominator: $\ln(x^2+3)+c$ (no modulus needed as $x^2+3>0$).

b. Write $\tan x$ as a fraction whose numerator is, up to a sign, the derivative of the denominator. $\tan x = \frac{\sin x}{\cos x} = -\frac{-\sin x}{\cos x}$, so $\int \tan x\,dx = -\ln|\cos x|+c$. This can also be written $\ln|\sec x|+c$.

c. The derivative of e^x+1 is e^x, so the integral is $\ln(e^x+1)$. $[\ln(e^x+1)]_0^1 = \ln(e+1)-\ln 2 = \ln\frac{e+1}{2}$. Numerically this is about 0.620.

Answer to Question 3

Method (substitution): choose u, find du in terms of dx, change the limits to values of u, rewrite the whole integral in u, then integrate. With new limits there is no need to go back to x.

a. $u = x^2 + 1$, $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$. Limits: $x = 0 \Rightarrow u = 1$, $x = 2 \Rightarrow u = 5$.
$\frac{1}{2}\int_1^5 u^3 du = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^5 = \frac{625 - 1}{8} = 78$. So the integral equals 78.

b. $u = x^2$, $du = 2x\,dx$. Limits 0 and 1. $\frac{1}{2}\int_0^1 e^u du = \frac{1}{2}(e - 1) = 0.859$. So the exact value is $\frac{1}{2}(e - 1) \approx 0.859$.

Answer to Question 4

Method (substitution): choose u, find du in terms of dx, change the limits to values of u, rewrite the whole integral in u, then integrate. With new limits there is no need to go back to x. Here the substitution also changes the x outside the root: $x = u - 1$.

a. $u = x + 1$, $x = u - 1$, $dx = du$. Limits: $x = 0 \Rightarrow u = 1$; $x = 3 \Rightarrow u = 4$.
$\int_1^4 (u-1)u^{1/2}du = \int_1^4 \left(u^{3/2} - u^{1/2}\right)du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 = \left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{116}{15} = 7.73$
.

Answer: $\int_0^3 x\sqrt{x+1}\,dx = \frac{116}{15} \approx 7.73$.

Answer to Question 5

Method (integration by parts): $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Choose u as the factor that becomes simpler when differentiated (a power of x), except with $\ln x$, which should always be u.

a. Choose $u = x$, because it becomes 1 when differentiated. $u = x$, $\frac{dv}{dx} = e^{2x}$, so $\frac{du}{dx} = 1$, $v = \frac{1}{2}e^{2x}$.
$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c = \frac{1}{4}e^{2x}(2x - 1) + c$. Check: differentiating $\frac{1}{4}e^{2x}(2x - 1)$ with the product rule gives $\frac{1}{2}e^{2x}(2x - 1) + \frac{1}{2}e^{2x} = xe^{2x}$.

b. We cannot integrate $\ln x$ directly, so choose $u = \ln x$. $u = \ln x$, $\frac{dv}{dx} = x$, $v = \frac{x^2}{2}$.
$\left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x}{2}dx = \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \frac{e^2-1}{4} = \frac{e^2 + 1}{4} = 2.10$. So the exact value is $\frac{e^2 + 1}{4} \approx 2.10$.

Answer to Question 6

Method: first split the fraction into partial fractions, using the cover-up values $x = 1$ and $x = -3$. Each partial fraction then integrates to a logarithm, and the laws of

logarithms combine the result.

Let $\frac{7x+1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$. At $x = 1$: $A = \frac{8}{4} = 2$. At $x = -3$: $B = \frac{-20}{-4} = 5$. Then

$\int_2^4 \left(\frac{2}{x-1} + \frac{5}{x+3}\right) dx = [2\ln|x-1| + 5\ln|x+3|]_2^4 = (2\ln 3 + 5\ln 7) - (0 + 5\ln 5) = 2\ln 3 + 5\ln\frac{7}{5} = 3.88$

.

Answer: $\int_2^4 \frac{7x+1}{(x-1)(x+3)} dx = 2\ln 3 + 5\ln\frac{7}{5}$, so $a = 2$ and $b = 5$.

Answer to Question 7

a. Substitute each value of t into R. $R(0) = 12$ litres per minute. $R(10) = 12e^{-2} = 1.62$ litres per minute. So the flow slows down a lot in 10 minutes.

b. The volume is the integral of the rate of flow. $V = \int_0^{10} 12e^{-0.2t} dt = \left[-60e^{-0.2t}\right]_0^{10} = 60(1 - e^{-2}) = 51.9$ litres. So about 51.9 litres flow in during the first 10 minutes.

c. Write the volume after T minutes as a formula and look at the exponential term. $V(T) = 60(1 - e^{-0.2T})$. Since $e^{-0.2T} > 0$, $V(T)$ for every T, and as $T \to \infty$, $V \to 60$. So the tank never receives more than 60 litres from this tap.

Answer to Question 8

a. Use the product rule and set the derivative to zero; $e^{-0.5t}$ is never zero. $\frac{dv}{dt} = 3e^{-0.5t}(1 - 0.5t) = 0$ at $t = 2$. $v(2) = 6e^{-1} = 2.21$ m s^{-1}. So the sprinter is fastest after 2 seconds, at about 2.21 m s^{-1}.

b. Distance is the integral of speed. Use parts with $u = t$. $\int_0^4 3te^{-0.5t} dt$: $u = t$, $\frac{dv}{dt} = e^{-0.5t}$, $v = -2e^{-0.5t}$.

$3\left\{\left[-2te^{-0.5t}\right]_0^4 + 2\int_0^4 e^{-0.5t} dt\right\} = 3\left\{-8e^{-2} + \left[-4e^{-0.5t}\right]_0^4\right\} = 3\{-8e^{-2} - 4e^{-2} + 4\} = 12 - 36e^{-2} = 7.13$ m. So the sprinter runs about 7.13 m in the first 4 seconds.

c. Average speed is total distance divided by total time. Average speed $= \frac{7.13}{4} = 1.78$ m s^{-1}.

Answer to Question 9

Method (separating the variables): put all the y terms with dy on one side and all the x (or t) terms on the other, integrate both sides, add one constant, then use the initial condition to find it.

a. Separate: $\int \frac{1}{\theta - 20} d\theta = \int -k\,dt$, so $\ln(\theta - 20) = -kt + c$ and $\theta - 20 = Ae^{-kt}$. At $t = 0$, $A = 70$, so $\theta = 20 + 70e^{-kt}$. This is the required solution.
b. Use the second condition $\theta = 55$ when $t = 10$. $55 = 20 + 70e^{-10k}$ gives $e^{-10k} = \frac{1}{2}$, so $k = \frac{\ln 2}{10} = 0.0693$. So $k = \frac{\ln 2}{10}$: the excess temperature halves every 10 minutes.
c. Substitute $\theta = 30$ and take logarithms. $30 = 20 + 70e^{-kt}$ gives $e^{-kt} = \frac{1}{7}$, so $t = \frac{\ln 7}{k} = \frac{10 \ln 7}{\ln 2} = 28.1$ minutes. So the coffee is drinkable after about 28 minutes.

Answer to Question 10

Method (separating the variables): put all the y terms with dy on one side and all the x (or t) terms on the other, integrate both sides, add one constant, then use the initial condition to find it.

a. $\int \frac{1}{P} dP = \int 0.04\,dt$, so $\ln P = 0.04t + c$, $P = Ae^{0.04t}$. With $P(0) = 500$, $P = 500e^{0.04t}$.
b. $500e^{0.04t} = 1000$ gives $t = \frac{\ln 2}{0.04} = 17.3$ years.
c. Food and space are limited, so growth cannot continue exponentially; predators, disease and hunting limit the size of the herd.

Answer to Question 11

Method (separating the variables): put all the y terms with dy on one side and all the x (or t) terms on the other, integrate both sides, add one constant, then use the initial condition to find it.

a. Write $\frac{1}{\sqrt{h}} = h^{-1/2}$ and integrate both sides. $\int h^{-1/2} dh = \int -k\,dt$ gives $2\sqrt{h} = -kt + c$. At $t = 0$: $2 \times 3 = c$, so $2\sqrt{h} = 6 - kt$. This is the required result.
b. Use $h = 4$ when $t = 5$. At $t = 5$: $2 \times 2 = 6 - 5k$, so $k = 0.4$.
c. Set $h = 0$ in the solution. $h = 0$: $0 = 6 - 0.4t$, so $t = 15$ minutes. So the tank is empty after 15 minutes.
d. After $t = 15$ the formula would give $\sqrt{h}$ and then a depth that increases again, which is impossible because the tank is empty.

Answer to Question 12

Method (integration by parts): $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$. Choose u as the factor that

becomes simpler when differentiated (a power of x), except with $\ln x$, which should always be u.

a. The number of tickets is the integral of the rate. Use parts with $u = t$ and $\frac{dv}{dt} = e^{-0.2t}$. $\int_0^{10} 40te^{-0.2t}dt$: $u = t$, $v = -5e^{-0.2t}$.
$40\left\{\left[-5te^{-0.2t}\right]_0^{10} + 5\int_0^{10} e^{-0.2t}dt\right\} = 40\{-50e^{-2} + [-25e^{-0.2t}]_0^{10}\} = 40\{25 - 75e^{-2}\} = 1000 - 3000e^{-2} \approx 594$ tickets. So about 594 tickets are sold in the first 10 days.
b. Let the upper limit tend to infinity: $te^{-0.2t} \to 0$ and $e^{-0.2t} \to 0$. Total over all time: $40\{0 + 25\} = 1000$, since $\int_0^{\infty} te^{-0.2t}dt = 25$. So at most 1000 tickets are ever sold. So sales level off at 1000 tickets and the 1200 seats are never all sold.
c. Differentiate S with the product rule and set the result to zero. $\frac{dS}{dt} = 40e^{-0.2t}(1 - 0.2t) = 0$ when $t = 5$ days. This agrees with the table, where the largest value is at $t = 5$.

Answer to Question 13

a. We cannot integrate $\sin^2 t$ directly, so use the double angle identity. $\int_0^{\pi} \frac{1 - \cos 2t}{2}dt = \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi} = \frac{\pi}{2}$. So $\int_0^{\pi} \sin^2 t\, dt = \frac{\pi}{2}$.
b. The mean value of a function over an interval is the integral divided by the length of the interval. Mean value $= \frac{1}{\pi} \times \frac{\pi}{2} = \frac{1}{2}$, so the mean power is $\frac{P_0}{2}$. So on average the circuit uses half of its peak power.

Answer to Question 14

Method (separating the variables): put all the y terms with dy on one side and all the x (or t) terms on the other, integrate both sides, add one constant, then use the initial condition to find it.

a. Separate: $\int \frac{dm}{0.5 - 0.02m} = \int dt$, so $-50\ln(0.5 - 0.02m) = t + c$. At $t = 0$, $m = 0$: $c = -50\ln 0.5$. So $\ln\left(\frac{0.5 - 0.02m}{0.5}\right) = -0.02t$, $1 - 0.04m = e^{-0.02t}$, and $m = 25\left(1 - e^{-0.02t}\right)$.
b. As $t \to \infty$, $m \to 25$ kg (when inflow equals outflow).
c. $25(1 - e^{-0.02t}) = 20$ gives $e^{-0.02t} = 0.2$, so $t = 50\ln 5 = 80.5$ days.

Answers: the mass approaches 25 kg, and it reaches 20 kg after about 80.5 days.

Answer to Question 15

a. $\dfrac{dy}{dx} = e^{-x}(1-x) = 0$ at $x = 1$: $y = e^{-1}$. The maximum is at $(1, e^{-1}) = (1, 0.368)$.

b. $u = x$, $\dfrac{dv}{dx} = e^{-x}$, $v = -e^{-x}$.
$\int_0^3 xe^{-x}dx = [-xe^{-x}]_0^3 + \int_0^3 e^{-x}dx = -3e^{-3} + [-e^{-x}]_0^3 = -3e^{-3} - e^{-3} + 1 = 1 - 4e^{-3}$
.

c. $1 - 4(0.04979) = 0.801$.

d. Ordinates at $x = 0, 1, 2, 3$: $0,\ 0.3679,\ 0.2707,\ 0.1494$. Area $\approx \frac{1}{2} \times 1 \times [0 + 0.1494 + 2(0.3679 + 0.2707)] = \frac{1}{2}(1.4266) = 0.713$. This underestimates the true value 0.801 because the curve is concave on part of the interval and the trapezia cut under the curve near the maximum.

Answer to Question 16

a. A slice at height x with thickness δx has volume approximately $A(x)\delta x = x^2\delta x$. Adding all slices from $x = 0$ to $x = 3$ gives the volume, and the approximation improves as $\delta x \to 0$.

b. $\int_0^3 x^2 dx = \left[\dfrac{x^3}{3}\right]_0^3 = 9$ m^3.

c. $9 \div 0.5 = 18$ minutes.

d. Left-hand ends $x = 0, 1, 2$: $0 + 1 + 4 = 5$ m^3. This is an underestimate because $A(x)$ is increasing, so each slice uses the smallest cross-section on its interval.

Answer to Question 17

a. $\dfrac{1}{5} \cdot \dfrac{(5-P)+P}{P(5-P)} = \dfrac{1}{5} \cdot \dfrac{5}{P(5-P)} = \dfrac{1}{P(5-P)}$.

b. $\int \dfrac{dP}{P(5-P)} = \int 0.02\,dt$ so $\dfrac{1}{5}[\ln P - \ln(5-P)] = 0.02t + c$, i.e.
$\ln \dfrac{P}{5-P} = 0.1t + c\prime$. At $t = 0$, $P = 1$: $c\prime = \ln \frac{1}{4}$. So $\dfrac{P}{5-P} = \dfrac{1}{4}e^{0.1t}$. Rearranging, $4P = 5e^{0.1t} - Pe^{0.1t}$, $P(4 + e^{0.1t}) = 5e^{0.1t}$, hence $P = \dfrac{5e^{0.1t}}{4 + e^{0.1t}}$.

c. $P = 4$: $\dfrac{P}{5-P} = 4 = \dfrac{1}{4}e^{0.1t}$, so $e^{0.1t} = 16$ and $t = 10 \ln 16 = 27.7$ years.

d. As $t \to \infty$, $P \to 5$ (5000 fish), the carrying capacity. Growth is nearly exponential when P is small and slows as P approaches 5, giving an S-shaped curve.

Answer to Question 18

Method (substitution): choose u, find du in terms of dx, change the limits to values of u , rewrite the whole integral in u, then integrate. With new limits there is no need to go

back to x.

a. $u = 4 - x^2$, $du = -2x\,dx$, so $x\,dx = -\frac{1}{2}du$.
$\int -\frac{1}{2}u^{-1/2}du = -u^{1/2} + c = -\sqrt{4 - x^2} + c$.

b. $\left[-\sqrt{4 - x^2}\right]_0^{1.5} = -\sqrt{1.75} + 2 = 0.677$ m^2.

c. $\left[-\sqrt{4 - x^2}\right]_0^{2} = 0 + 2 = 2$. At $x = 2$ the integrand is undefined (the curve has a vertical asymptote), so this is an improper integral, but the area is finite.

Answer to Question 19

Method (integration by parts): $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Choose u as the factor that becomes simpler when differentiated (a power of x), except with $\ln x$, which should always be u.

a. First: $u = t^2$, $v = -e^{-t}$: $\int t^2 e^{-t}dt = -t^2e^{-t} + 2\int te^{-t}dt$. Second: $\int te^{-t}dt = -te^{-t} + \int e^{-t}dt = -te^{-t} - e^{-t}$. Combine: $-t^2e^{-t} - 2te^{-t} - 2e^{-t} = -e^{-t}(t^2 + 2t + 2)$.

b. $Q = \left[-e^{-t}(t^2 + 2t + 2)\right]_0^3 = 2 - 17e^{-3} = 1.15$ C.

c. As $T \to \infty$, $e^{-T}(T^2 + 2T + 2) \to 0$, so $Q \to 2$ C. The current dies away so the charge approaches a finite limit.

Answer to Question 20

Method (integration by parts): $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Choose u as the factor that becomes simpler when differentiated (a power of x), except with $\ln x$, which should always be u. Here we use it twice and the original integral reappears, so we solve an equation for J.

a. First: $J = [e^x \sin x]_0^{\pi/2} - \int_0^{\pi/2} e^x \cos x\,dx = e^{\pi/2} - K$ where $K = \int_0^{\pi/2} e^x \cos x\,dx$.
Second: $K = [e^x \cos x]_0^{\pi/2} + \int_0^{\pi/2} e^x \sin x\,dx = (0 - 1) + J$. So $J = e^{\pi/2} - (J - 1)$, giving $2J = e^{\pi/2} + 1$.

b. $J = \dfrac{4.810 + 1}{2} = 2.905$.

Answer: $J = \dfrac{e^{\pi/2} + 1}{2} \approx 2.91$.

Answer to Question 21

Method: when an integral cannot be found directly, look for a related integral; here adding K and L gives something simple, and a symmetry substitution shows they are

equal.

a. $K + L = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$.

b. Let $u = \frac{\pi}{2} - x$, $dx = -du$. Limits: $x = 0 \Rightarrow u = \frac{\pi}{2}$; $x = \frac{\pi}{2} \Rightarrow u = 0$. $\sin x = \cos u$ and $\cos x = \sin u$. $K = \int_{\pi/2}^{0} \frac{\cos u}{\cos u + \sin u}(-du) = \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u} du = L$.

c. $2K = \frac{\pi}{2}$ so $K = \frac{\pi}{4}$.