Test answers - Further trigonometry

Year: Year 13 · Duration: 40 minutes · /20

Answer to Question 1 (4 marks)

Method: the small-angle approximations only work in radians: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$, $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$. Replace each function, simplify, and the powers of θ usually cancel.

Mark scheme: 1 mark for each approximation in part a), 1 mark for the value and 1 mark for the percentage in part b).

a. $\theta\tan 2\theta \approx 2\theta^2$ and $1-\cos 3\theta \approx \dfrac{9\theta^2}{2}$, so the expression $\approx \dfrac{2\theta^2}{9\theta^2/2} = \dfrac{4}{9}$.

b. $\dfrac{0.1\tan 0.2}{1-\cos 0.3} = \dfrac{0.020271}{0.044664} = 0.4539$. Difference from $\frac{4}{9} = 0.4444$: $\dfrac{0.4539 - 0.4444}{0.4539} = 2.1\%$.

Answer: the expression is approximately $\frac{4}{9}$, and for $\theta = 0.1$ the true value differs from it by about 2.1%.

Answer to Question 2 (4 marks)

Method: replace $\tan^2 x$ by $\sec^2 x - 1$ to get a quadratic in $\sec x$, factorise, then change each $\sec x$ value into a cosine.

a. Use $\tan^2 x = \sec^2 x - 1$: $2\sec^2 x - 2 + \sec x = 1$, so $2\sec^2 x + \sec x - 3 = 0$ [1 mark]. Factorise: $(2\sec x + 3)(\sec x - 1) = 0$ [1 mark]. $\sec x = 1$ gives $x = 0^\circ$. $\sec x = -\frac{3}{2}$ gives $\cos x = -\frac{2}{3}$, so $x = 131.8^\circ$ or 228.2° [2 marks]. Solutions: **0°, 131.8°, 228.2°**.

Check $x = 0^\circ$: $2 \times 0 + 1 = 1$ ✓. The value 360° is excluded by the strict inequality in the interval.

Answer to Question 3 (4 marks)

Method: use the compound angle formulae $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$, $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ and $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$, with exact values of 30°, 45° and 60°. Mark scheme: 2 marks for each part.

a. Write $105^\circ = 60^\circ + 45^\circ$.

$\sin 105^\circ = \sin(60^\circ + 45^\circ) = \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} + \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$. So

$\sin 105^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4} \approx 0.966$.

b. Both angles are acute, so $\cos A$ and $\sin B$ are positive; use Pythagoras to find them. $\cos A = \dfrac{12}{13}$, $\sin B = \dfrac{3}{5}$.

$\cos(A-B) = \cos A \cos B + \sin A \sin B = \dfrac{12}{13}\cdot\dfrac{4}{5} + \dfrac{5}{13}\cdot\dfrac{3}{5} = \dfrac{48+15}{65} = \dfrac{63}{65}$. So $\cos(A-B) = \dfrac{63}{65}$.

Answer to Question 4 (4 marks)

Method: replace $\cos 2x$ by the form that only contains $\sin x$, so that the equation becomes a quadratic in $\sin x$.

Use $\cos 2x = 1 - 2\sin^2 x$: $2 - 4\sin^2 x + 3\sin x = 1$, so $4\sin^2 x - 3\sin x - 1 = 0$ [2 marks]. Factorise: $(4\sin x + 1)(\sin x - 1) = 0$ [1 mark]. $\sin x = 1$ gives $x = 90^\circ$. $\sin x = -\frac{1}{4}$ gives $x = 194.5^\circ$ or 345.5° [1 mark]. Solutions: **90°, 194.5°, 345.5°**.

Check $x = 90^\circ$: $2\cos 180^\circ + 3 = -2 + 3 = 1$ ✓. For $\sin x = -\frac{1}{4}$, the principal value is -14.5°, so the answers in range are $180^\circ + 14.5^\circ$ and $360^\circ - 14.5^\circ$.

Answer to Question 5 (4 marks)

Method: expand $R\cos(x-\alpha)$, compare coefficients of $\cos x$ and $\sin x$, then use $R = \sqrt{a^2+b^2}$ and $\tan\alpha = \dfrac{b}{a}$. Mark scheme: 1 mark for R, 1 mark for α, 2 marks for the solutions.

a. Compare $R\cos\alpha = 8$ and $R\sin\alpha = 15$. $R = \sqrt{64+225} = 17$.

$R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$, so $\tan\alpha = \dfrac{15}{8}$, $\alpha = 61.93^\circ$. Expression $= 17\cos(x - 61.93^\circ)$. Check: $17\cos 61.93^\circ = 8.00$.

b. Divide by 17 and use the inverse cosine; cosine is positive in the first and fourth quadrants. $\cos(x - 61.93^\circ) = \dfrac{10}{17} = 0.5882$. $x - 61.93^\circ = \pm 53.97^\circ$, so $x = 115.9^\circ$ or $x = 7.96^\circ$. Solutions: **8.0°, 115.9°**. Both values lie in the required interval.