Lesson - Further trigonometry

Year: Year 13

This chapter extends trigonometry with small-angle approximations, the reciprocal functions, compound and double angle formulae and the R form, then uses them to prove identities and solve equations.

Starter activity

Use a calculator in radian mode to compare $\sin 0.1$ with 0.1, and $\sin 1$ with 1. What do you notice? Then draw the graphs of $y = \sin x$ and $y = x$ on the same axes for x between 0 and 1. Why are the graphs almost the same near the origin?

Small-angle approximations

Rule: small-angle approximations

When θ is small **and in radians**: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$ and $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$. For $\theta = 0.05$, $\sin 0.05 = 0.049979 \approx 0.05$; for $\theta = 0.1$, $\cos 0.1 = 0.995004$ and the approximation gives 0.995.

Worked example 1

Find an approximation for $\dfrac{1-\cos 2\theta}{\theta \tan 3\theta}$ when θ is small.

$1 - \cos 2\theta \approx \dfrac{(2\theta)^2}{2} = 2\theta^2$ and $\theta \tan 3\theta \approx 3\theta^2$, so the ratio is approximately $\frac{2}{3}$.

Watch out

Check that your calculator is in radian mode, and never use these approximations in degrees unless you convert first.

Reciprocal functions: sec, cosec and cot

Definition: sec, cosec and cot

$\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$. They are undefined where the denominator is zero, giving vertical asymptotes: $\sec\theta$ at $90^\circ, 270^\circ, \ldots$ and $\operatorname{cosec}\theta$ at $0^\circ, 180^\circ, \ldots$. The graphs of sec and cosec never lie between -1 and 1.

Rule: new identities

Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ or $\sin^2\theta$ gives

$$1 + \tan^2\theta \equiv \sec^2\theta, \qquad 1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta.$$

Use them to turn a mixed equation into a quadratic in one function.

Worked example 2

Solve $\sec x = 3$ and $\operatorname{cosec}^2 x - \cot x - 3 = 0$ for $0^\circ \le x$.

$\cos x = \frac{1}{3}$, so $x = 70.5^\circ$ or 289.5°. For the second, replace $\operatorname{cosec}^2 x$ by $1 + \cot^2 x$: $\cot^2 x - \cot x - 2 = 0$, so $\cot x = 2$ or -1, i.e. $\tan x = \frac{1}{2}$ or -1, giving $x = 26.6^\circ, 206.6^\circ, 135^\circ, 315^\circ$.

Compound angle formulae

Rule: addition formulae

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$, $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$, $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$. Note the sign change in the cosine formula.

Worked example 3

Find the exact value of $\sin 75^\circ$.

$$\sin(45^\circ + 30^\circ) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Worked example 4

$\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$ with both angles acute. Find $\sin(A+B)$.

$\cos A = \frac{4}{5}$, $\sin B = \frac{12}{13}$, so $\sin(A+B) = \frac{3}{5}\cdot\frac{5}{13} + \frac{4}{5}\cdot\frac{12}{13} = \frac{63}{65}$.

Double angle formulae

Rule: double angle formulae

Setting $B = A$: $\sin 2A = 2\sin A\cos A$, $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$, $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$. Choose the form of $\cos 2A$ that matches the other function in the equation.

Worked example 5

Solve $\cos 2x = \sin x$ for $0^\circ \leq x$.

$1 - 2\sin^2 x = \sin x$, so $2\sin^2 x + \sin x - 1 = 0$, $(2\sin x - 1)(\sin x + 1) = 0$, and $x = 30^\circ, 150^\circ, 270^\circ$.

Worked example 6

Prove that $\dfrac{\sin 2x}{1 + \cos 2x} \equiv \tan x$.

$$\frac{2\sin x \cos x}{1 + (2\cos^2 x - 1)} = \frac{2\sin x \cos x}{2\cos^2 x} = \tan x. \blacksquare$$

The R form: $a\sin x + b\cos x$

Rule: the R form

For $a, b > 0$: $a\sin x + b\cos x = R\sin(x + \alpha)$ with $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \dfrac{b}{a}$ (equally $R\cos(x - \beta)$). The maximum is R and the minimum $-R$.

Method: solving $a\sin\theta + b\cos\theta = c$

1. Write $a\sin\theta + b\cos\theta$ as $R\sin(\theta + \alpha)$.
2. Solve $\sin(\theta + \alpha) = \dfrac{c}{R}$ for the whole argument, using the extended range.
3. Subtract α and keep the answers in the range.

Worked example 7

Write $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$ and solve $3\sin\theta + 4\cos\theta = 2$ for $0^\circ \leq \theta$.

$R = 5$, $\tan\alpha = \frac{4}{3}$, so $\alpha = 53.13^\circ$ and the expression is $5\sin(x + 53.13^\circ)$, with maximum 5 at $x = 36.87^\circ$. Then $\sin(\theta + 53.13^\circ) = 0.4$, so $\theta + 53.13^\circ = 23.58^\circ$ or 156.42° (add 360° if needed): $\theta = 103.3^\circ$ or 330.4°. The expression $10 + 3\sin x + 4\cos x$ lies between 5 and 15.

Proof and problem solving

Worked example 8

Prove that $\cos 3x \equiv 4\cos^3 x - 3\cos x$.

$\cos 3x = \cos(2x + x) = \cos 2x\cos x - \sin 2x\sin x = (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x = 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x = 4\cos^3 x - 3\cos x$

. ■

Method: choosing the right identity

1. $\cos 2x$ with $\sin x$: use $1 - 2\sin^2 x$. $\cos 2x$ with $\cos x$: use $2\cos^2 x - 1$.
2. $\sec^2 x$ with $\tan x$: replace $\sec^2 x$ by $1 + \tan^2 x$. $\operatorname{cosec}^2 x$ with $\cot x$: use $1 + \cot^2 x$.
3. $\sin x$ and $\cos x$ to the first power: use the R form.
4. $\sin 2x$ with $\sin x$ or $\cos x$: expand and factorise, never divide.

Note: ranges and proofs

If the argument is $2x$ and $0^\circ \leq x$, then $0^\circ \leq 2x$: two full turns, so twice as many solutions. In a proof start from one side (often the more complicated), show each line, never move terms across the equals sign, and finish with exactly the expression in the question. Stuck? Convert everything to sine and cosine.

Watch out

$\sec x = \dfrac{1}{\cos x}$, not $\dfrac{1}{\sin x}$. Mind the sign in the cosine compound formula. Never divide by $\sin x$ or $\tan x$: factorise to keep every solution. Give all solutions in the range.

Why small-angle approximations work

Note: how accurate are they?

Use a calculator in radian mode. For $\theta = 0.5$: $\sin 0.5 = 0.4794$, $\tan 0.5 = 0.5463$ and $\cos 0.5 = 0.8776$, while the approximations give 0.5, 0.5 and 0.875. For $\theta = 0.1$: $\sin 0.1 = 0.09983$, $\tan 0.1 = 0.10033$ and $\cos 0.1 = 0.99500$ against 0.1, 0.1 and 0.995. The smaller θ is, the better the match. These come from the Maclaurin series you meet in Further Maths, where $\sin\theta = \theta - \dfrac{\theta^3}{6} + \cdots$ and $\cos\theta = 1 - \dfrac{\theta^2}{2} + \cdots$.

Worked example 9: a limit by small angles

Find an approximation for $\dfrac{\tan 5\theta \sin 2\theta}{\theta(1 - \cos\theta)}$ for small θ and explain what happens as $\theta \to 0$.

$\tan 5\theta \approx 5\theta$, $\sin 2\theta \approx 2\theta$ and $1 - \cos\theta \approx \dfrac{\theta^2}{2}$. The expression is approximately $\dfrac{5\theta \cdot 2\theta}{\theta \cdot \theta^2/2} = \dfrac{10\theta^2}{\theta^3/2} = \dfrac{20}{\theta}$, which grows without bound as $\theta \to 0$: there is no finite limit.

Identities and proof

Method: proving an identity

1. Start with the more complicated side (usually the left).
2. Rewrite everything in terms of $\sin$ and $\cos$, or use a double angle formula that matches the target.
3. Combine fractions and cancel, but never move terms across the identity sign.
4. Finish with the expression on the other side and end with a conclusion such as ■.

Worked example 10: proving with reciprocal functions

Prove that $\sec x + \tan x \equiv \dfrac{1+\sin x}{\cos x}$ and hence that $\dfrac{1}{\sec x - \tan x} \equiv \sec x + \tan x$.

$\sec x + \tan x = \dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x} = \dfrac{1+\sin x}{\cos x}$. For the second part, $(\sec x - \tan x)(\sec x + \tan x) = \sec^2 x - \tan^2 x = 1$, so $\dfrac{1}{\sec x - \tan x} = \sec x + \tan x$. ■

Worked example 11: a double angle identity

Prove that $\dfrac{1-\cos 2x}{\sin 2x} \equiv \tan x$.

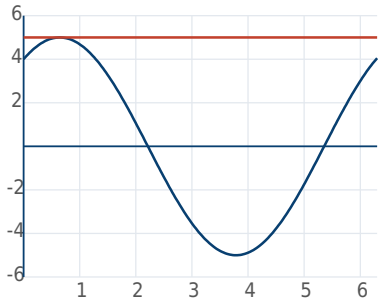
$1 - \cos 2x = 2\sin^2 x$ and $\sin 2x = 2\sin x \cos x$, so the expression is $\dfrac{2\sin^2 x}{2\sin x \cos x} = \dfrac{\sin x}{\cos x} = \tan x$. ■

Worked example 12: exact value with tan

Given that $\tan A = \dfrac{3}{4}$ and A is acute, find $\sin 2A$, $\cos 2A$ and $\tan 2A$.

The triangle is 3, 4, 5, so $\sin A = \dfrac{3}{5}$ and $\cos A = \dfrac{4}{5}$. $\sin 2A = 2 \cdot \dfrac{3}{5} \cdot \dfrac{4}{5} = \dfrac{24}{25}$. $\cos 2A = \dfrac{16}{25} - \dfrac{9}{25} = \dfrac{7}{25}$. $\tan 2A = \dfrac{24}{7}$. Check with the formula $\dfrac{2\tan A}{1-\tan^2 A} = \dfrac{3/2}{7/16} = \dfrac{24}{7}$.

Using the combined form in context



Note: reading the graph

The curve $y = 3\sin x + 4\cos x$ is a wave of amplitude 5, shifted to the left by $\alpha = 0.927$ radians ($53.13°$). It touches the red line $y = 5$ at its maximum and crosses the x-axis at $x = 2.21$ and $x = 5.36$. The R form turns two terms into one wave, which makes maxima, minima and solutions much easier to find.

Worked example 13: R form with the cosine

Write $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x - \alpha)$ and solve $\sin x - \sqrt{3}\cos x = 1$ for $0° \leq x$.

$R = \sqrt{1 + 3} = 2$ and $\tan\alpha = \sqrt{3}$, so $\alpha = 60°$ and the expression is $2\sin(x - 60°)$. Then $\sin(x - 60°) = \frac{1}{2}$, so $x - 60° = 30°$ or $150°$, giving $x = 90°$ or $210°$.

Worked example 14: range of a function with the R form

Find the greatest and least values of $\dfrac{12}{5 + 3\sin x + 4\cos x}$.

The denominator is $5 + 5\sin(x + 53.13°)$, which lies between 0 and 10. The value 0 is reached when the sine is -1, so the expression is not defined there. Close to that point it becomes very large, so there is no greatest value; the least value is $\dfrac{12}{10} = 1.2$ when the sine is $+1$.

Exact values worth knowing

Key fact: exact values

You are expected to know these without a calculator. $\sin 30° = \cos 60° = \frac{1}{2}$; $\sin 45° = \cos 45° = \dfrac{\sqrt{2}}{2}$; $\sin 60° = \cos 30° = \dfrac{\sqrt{3}}{2}$; $\tan 30° = \dfrac{1}{\sqrt{3}}$, $\tan 45° = 1$, $\tan 60° = \sqrt{3}$. The reciprocal values follow at once, for example $\sec 30° = \dfrac{2}{\sqrt{3}}$ and $\cot 60° = \dfrac{1}{\sqrt{3}}$. With the compound angle formulae these produce the exact values of

15°, 75° and 105°, which appear in many questions.

Solving equations in an extended range

Worked example 15: an equation with a double argument

Solve $\sin 2x = 0.6$ **for** $0^\circ \leq x$**.**

Let $u = 2x$, which ranges over $0^\circ \leq u$. $\sin u = 0.6$ gives $u = 36.87^\circ$, 143.13°, 396.87°, 503.13°. Halving: $x = 18.4^\circ$, 71.6°, 198.4°, 251.6°. There are four solutions because the argument covers two full turns. The rule is: double the range, solve, then divide.

Worked example 16: using a compound angle in an equation

Solve $\cos(x + 30^\circ) = \sin x$ **for** $0^\circ \leq x$**.**

Expand: $\cos x \cos 30^\circ - \sin x \sin 30^\circ = \sin x$, so $\dfrac{\sqrt{3}}{2}\cos x = \frac{3}{2}\sin x$, giving

$\tan x = \dfrac{\sqrt{3}/2}{3/2} = \dfrac{1}{\sqrt{3}}$. So $x = 30^\circ$ or 210°. Check $x = 30^\circ$: $\cos 60^\circ = 0.5 = \sin 30^\circ$.

Note: choosing the method

Ask yourself three questions before you start solving. First, how many different trigonometric functions are there? If more than one, use an identity to reduce them to one. Second, are the arguments the same? If you see both x and $2x$, apply a double angle formula. Third, does the equation have the form $a\sin x + b\cos x = c$? If so, use the R form, or divide by $\cos x$ and use $\tan x$ if you are sure $\cos x$ is never zero in the range. Writing down which tool you chose and why earns method marks even if the arithmetic goes wrong.

Worked example 17: rewriting and then solving

Solve $3\cos 2x - 5\cos x + 2 = 0$ **for** $0^\circ \leq x$**.**

Use $\cos 2x = 2\cos^2 x - 1$: $6\cos^2 x - 3 - 5\cos x + 2 = 0$, so $6\cos^2 x - 5\cos x - 1 = 0$. This factorises as $(6\cos x + 1)(\cos x - 1) = 0$. $\cos x = 1$ gives $x = 0^\circ$. $\cos x = -\frac{1}{6}$ gives $x = 99.6^\circ$ or 260.4°. Solutions: 0°, 99.6°, 260.4°.

More on combining sine and cosine

Method: finding R and the angle

1. Write $a\sin x + b\cos x = R\sin(x + \alpha)$ and expand the right-hand side as $R\cos\alpha\sin x + R\sin\alpha\cos x$.

2. Match coefficients: $R\cos\alpha = a$ and $R\sin\alpha = b$.
3. Square and add to get $R = \sqrt{a^2 + b^2}$, then divide to get $\tan\alpha = \dfrac{b}{a}$.
4. Check the quadrant of α: both $R\cos\alpha$ and $R\sin\alpha$ must have the correct signs.

Worked example 18: R form with a negative coefficient

Write $2\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$ with α acute, and find its least value.

$R\sin(x-\alpha) = R\sin x\cos\alpha - R\cos x\sin\alpha$, so $R\cos\alpha = 2$ and $R\sin\alpha = 5$. Then $R = \sqrt{29} = 5.385$ and $\tan\alpha = \dfrac{5}{2}$, so $\alpha = 68.20°$. The expression is $\sqrt{29}\sin(x - 68.20°)$, whose least value is $-\sqrt{29} = -5.39$. It occurs when $x - 68.20° = 270°$, that is $x = 338.2°$.

Worked example 19: tan of a sum in exact form

Given that $\tan A = 2$ and $\tan B = 3$, find $\tan(A + B)$ and explain the sign of the answer.

$\tan(A+B) = \dfrac{2+3}{1-2\times 3} = \dfrac{5}{-5} = -1$. Both angles are acute, since their tangents are positive, and $A = 63.4°$, $B = 71.6°$. The sum is $135°$, an obtuse angle, where tangent is negative. Always check that the sign of the result agrees with the size of the angle.

Watch out: errors seen in examinations

- Dividing both sides of an equation by $\sin x$ or $\cos x$ and losing solutions: factorise instead.
- Using degrees in a small-angle approximation.
- Writing $\sin(A + B) = \sin A + \sin B$. This is false: check with $A = B = 30°$.
- Giving only the principal value after solving $\sin(x + \alpha) = k$; list all values in the extended range, then subtract α.

Did you know?

The compound angle formulae were known to the Greek astronomer Ptolemy about 1900 years ago. He used a version of $\sin(A - B)$ to build a table of chord lengths so that navigators could find the position of stars. Today the same formulae sit inside the signal processing used in every mobile phone.

Otto's tip

Before you solve, say how many solutions you expect in the range. It stops you missing the second angle.