Worksheet - Further trigonometry

Year: Year 13 · 20

Show your working.

Question 1

Question 1 – Small-angle approximations *(Section A · Skills practice)*

In this question θ is small and measured in radians. Use the approximations $\sin\theta \approx \theta$, $\tan\theta \approx \theta$ and $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$.

a. Estimate $\cos 0.06$ and compare with your calculator. [2 marks]

b. Find an approximation for $\dfrac{1-\cos 4\theta}{\theta \sin 2\theta}$. [3 marks]

c. Find an approximation for $\dfrac{\sin 3\theta \tan 2\theta}{\theta^2}$. [3 marks]

Question 2

Question 2 – Exact values of reciprocal functions *(Section A · Skills practice)*

Write down the exact value of each expression. Use a right-angled triangle or the unit circle where necessary.

a. $\sec 60^\circ$ and $\operatorname{cosec} 30^\circ$ [2 marks]

b. $\cot 45^\circ$ and $\cot 150^\circ$ [3 marks]

c. $\sec 120^\circ$ and $\operatorname{cosec} 240^\circ$ [3 marks]

Question 3

Question 3 – Equations with sec and tan *(Section A · Skills practice)*

Solve each equation for $0^\circ \leq x$, giving answers to 1 decimal place where necessary.

a. $\sec x = -2$ [2 marks]

b. $\tan^2 x + \sec x = 1$ [5 marks]

c. $\operatorname{cosec}^2 x - \cot x = 3$ [5 marks]

Question 4

Question 4 – Compound angle formulae *(Section A · Skills practice)*

Use the compound angle formulae to find the exact value of each expression.

a. $\sin 15^\circ$, using $15^\circ = 45^\circ - 30^\circ$ [3 marks]

b. $\cos 75^\circ$ [3 marks]
c. $\tan 105^\circ$, using $105^\circ = 60^\circ + 45^\circ$ [4 marks]

Question 5

Question 5 – Double angle equations *(Section A · Skills practice)*

Solve each equation for $0^\circ \leq x$.

a. $\sin 2x = \sin x$ [4 marks]
b. $\cos 2x + 3\cos x + 2 = 0$ [5 marks]

Question 6

Question 6 – Combining sine and cosine *(Section A · Skills practice)*

$f(x) = 5\sin x + 12\cos x$.

a. Express $f(x)$ in the form $R\sin(x + \alpha)$ where $R > 0$ and 0°. Give α to 2 decimal places. [3 marks]
b. State the maximum and minimum values of $f(x)$ and the smallest positive value of x where the maximum occurs. [3 marks]
c. Solve $f(x) = 6.5$ for $0^\circ \leq x$. [4 marks]

Question 7

Question 7 – The wind turbine tower *(Section B · Real-life practice)*

A surveyor stands 400 m from the base of a wind turbine tower on flat ground. She measures the angle of elevation of the top as 0.08 radians.

a. Use a small-angle approximation to estimate the height of the tower. [2 marks]
b. Find the exact height using $h = 400\tan 0.08$, to 3 significant figures. [2 marks]
c. Find the percentage error in your estimate. [2 marks]
d. Explain why the approximation would be unsuitable for an angle of 1 radian. [2 marks]

Question 8

Question 8 – Building a ramp *(Section B · Real-life practice)*

A ramp must lift a wheelchair user a vertical height of $h = 0.45$ m up to a library entrance. The ramp makes an angle θ with the horizontal. The ramp length is $L = h\,\mathrm{cosec}\,\theta$ and the horizontal run is $d = h\cot\theta$.

a. Find L and d when $\theta = 5^\circ$. [3 marks]
b. The building regulations in the UK allow a gradient of at most 1 in 12, so $\tan\theta \leq \frac{1}{12}$. Find the largest allowed angle. [2 marks]
c. Hence find the minimum length of the ramp and the horizontal run needed. [3

marks]

Question 9

Question 9 – Tide times at the harbour *(Section B · Real-life practice)*

The depth of water in a harbour in Cornwall is modelled by $h = 6 + 3\sin(30t)^\circ + 4\cos(30t)^\circ$ metres, where t is the time in hours after midnight, for $0 \le t \le 12$.

a. Express $3\sin(30t)^\circ + 4\cos(30t)^\circ$ in the form $R\sin(30t + \alpha)^\circ$. [3 marks]
b. State the greatest and least depths. [2 marks]
c. Boats can only enter when the depth is at least 9 m. Find the times when this is possible. [5 marks]

Question 10

Question 10 – The football kick *(Section B · Real-life practice)*

A goalkeeper kicks a football at speed $u = 20$ m s^{-1} at an angle θ above the horizontal. Ignoring air resistance, the horizontal range is $R = \dfrac{u^2 \sin 2\theta}{g}$, taking $g = 9.8$ m s^{-2}.

a. Find the maximum range and the angle at which it occurs. [3 marks]
b. The ball must land 30 m away on a flat pitch. Show that $\sin 2\theta = 0.735$. [2 marks]
c. Find the two possible angles of projection. [3 marks]

Question 11

Question 11 – Joining two skate ramps *(Section B · Real-life practice)*

Two skateboard ramps are joined end to end. The first has a gradient of $\dfrac{1}{3}$ (so $\tan A = \dfrac{1}{3}$) and the second a gradient of $\dfrac{1}{2}$ (so $\tan B = \dfrac{1}{2}$), where A and B are acute angles.

a. Show that $\tan(A + B) = 1$. [3 marks]
b. Hence find the total angle $A + B$ through which the path has turned upwards. [1 mark]
c. Find $\sin(A + B)$ exactly. [3 marks]
d. Find $\tan(B - A)$. [2 marks]

Question 12

Question 12 – The stadium floodlight *(Section B · Real-life practice)*

A floodlight is mounted 12 m above a flat training pitch. The beam makes an angle θ

with the horizontal. The beam length is $12\,\text{cosec}\,\theta$ metres and the distance from the base of the pole to the lit spot is $12\cot\theta$ metres.

a. Find the beam length and the lit distance when $\theta = 30°$. [3 marks]
b. The groundskeeper wants to light a spot exactly 20 m from the pole. Find θ. [2 marks]
c. Show that $\text{cosec}^2\,\theta - \cot^2\theta = 1$ and use it to find the beam length in part (b) without finding θ. [4 marks]

Question 13

Question 13 – A playground swing *(Section B · Real-life practice)*

A swing has chains of length $L = 2.5$ m. When the chains make a small angle θ with the vertical, the seat is displaced horizontally by $L\sin\theta$ and raised by $L(1 - \cos\theta)$.

a. Use small-angle approximations to show that the horizontal displacement is approximately $L\theta$ and the rise is approximately $\dfrac{L\theta^2}{2}$. [2 marks]
b. For $\theta = 0.2$ radians find the approximate values of both quantities. [2 marks]
c. Find the exact values and the percentage errors, to 2 significant figures. [4 marks]

Question 14

Question 14 – Polarising filters *(Section B · Real-life practice)*

In a physics experiment light passes through a polarising filter. The intensity transmitted is $I = I_0\cos^2\theta$, where θ is the angle between the filter and the polarisation of the light, $0° \le \theta \le 90°$.

a. Show that $I = \dfrac{I_0}{2}(1 + \cos 2\theta)$. [2 marks]
b. Find θ when the intensity is $0.75I_0$. [2 marks]
c. Find θ when the intensity is $0.2I_0$, to 1 decimal place. [2 marks]
d. Explain why I never exceeds I_0. [1 mark]

Question 15

Question 15 – Proving triple-angle identities *(Section C · Problem solving)*

Triple-angle formulae are used in engineering to analyse waves. You may use $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 1 - 2\sin^2 x$.

a. Prove that $\text{cosec}\,2x + \cot 2x \equiv \cot x$. [4 marks]
b. Prove that $\sin 3x \equiv 3\sin x - 4\sin^3 x$. [4 marks]
c. Hence solve $\sin 3x = \sin x$ for $0° \le x \le 180°$. [4 marks]
d. Use part (a) with $x = 15°$ to show that $\cot 15° = 2 + \sqrt{3}$. [3 marks]

Question 16

Question 16 – Temperature through the day *(Section C · Problem solving)*

A weather station records the temperature T (°C) t hours after 11:00. A scientist models it as $T = 14 + 3\sin(15t)^\circ + 4\cos(15t)^\circ$, for $0 \le t \le 24$.

t (hours)	0	3	6	12	18
T (°C)	18.0	19.0	17.0	10.0	11.0

a. Verify two of the table values. [2 marks]
b. Write $3\sin(15t)^\circ + 4\cos(15t)^\circ$ in the form $R\sin(15t + \alpha)^\circ$ and state the maximum and minimum temperature. [4 marks]
c. At what time of day is the temperature greatest? [3 marks]
d. For how long in the 24 hours is the temperature at least 17 °C? [4 marks]

Question 17

Question 17 – A harder trigonometric equation *(Section C · Problem solving)*

\(2\sec^2 x + 3\tan x - 5 = 0, \qquad 0^\circ \le x\)

x	0°	30°	45°	60°
$\sec x$	1	1.155	1.414	2

a. Show that the equation can be written as $2\tan^2 x + 3\tan x - 3 = 0$. [2 marks]
b. Solve the equation, giving the solutions to 1 decimal place. [5 marks]
c. Explain why $\sec x$ never takes values between -1 and 1, using the table and the definition. [2 marks]
d. State how many solutions you expect for $\tan x = k$ in the given range, and why. [2 marks]

Question 18

Question 18 – Two angles in a roof truss *(Section C · Problem solving)*

A roof truss has two rafters making acute angles A and B with the horizontal, where $\sin A = \dfrac{8}{17}$ and $\cos B = \dfrac{3}{5}$.

a. Find $\cos A$ and $\sin B$ exactly. [2 marks]
b. Show that $\sin(A + B) = \dfrac{84}{85}$. [3 marks]
c. Find $\cos(A + B)$ and verify that $\sin^2(A + B) + \cos^2(A + B) = 1$. [3 marks]
d. Find $\tan(A - B)$ exactly. [4 marks]
e. Find $A + B$ in degrees to 1 decimal place and say whether the combined angle is acute. [2 marks]

Question 19

Question 19 – Brightness of a solar farm display *(Section C · Problem solving)*

The brightness reading of a monitor beside a solar farm is modelled by $B(x) = \dfrac{1}{7 + 3\sin x + 4\cos x}$ where x is the angle in degrees of the sun through the day, $0^\circ \le x$.

a. Write $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$. [3 marks]
b. Find the maximum and minimum values of B. [3 marks]
c. Find the value of x at which B is greatest. [2 marks]
d. Solve $B(x) = \dfrac{1}{4}$. [4 marks]

Question 20

Question 20 – An exact value for cos 36 degrees *(Section D · Challenge)*

Let $\theta = 36^\circ$ and $c = \cos\theta$.

a. Explain why $3\theta + 2\theta = 180^\circ$ and deduce that $\cos 3\theta = -\cos 2\theta$. [2 marks]
b. Using $\cos 3\theta = 4c^3 - 3c$ and $\cos 2\theta = 2c^2 - 1$, show that $4c^3 + 2c^2 - 3c - 1 = 0$. [2 marks]
c. Show that $(c + 1)$ is a factor and hence find the exact value of $\cos 36^\circ$. [4 marks]
d. Find the exact value of $\cos 72^\circ$ and show that $\cos 36^\circ - \cos 72^\circ = \dfrac{1}{2}$. [3 marks]