Answers - Further trigonometry

Year: Year 13

Answer to Question 1

a. $\cos 0.06 \approx 1 - \dfrac{0.0036}{2} = 1 - 0.0018 = 0.9982$. The calculator gives 0.998201, so the approximation is accurate to 4 decimal places.

b. $1 - \cos 4\theta \approx \dfrac{(4\theta)^2}{2} = 8\theta^2$ and $\theta \sin 2\theta \approx \theta \cdot 2\theta = 2\theta^2$. The ratio is approximately $\dfrac{8\theta^2}{2\theta^2} = \mathbf{4}$.

c. $\sin 3\theta \approx 3\theta$ and $\tan 2\theta \approx 2\theta$, so the numerator is $\approx 6\theta^2$ and the ratio is approximately **6**.

Answer to Question 2

Method: $\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$. Find the sine, cosine or tangent first (with its sign from the quadrant), then take the reciprocal.

a. Use $\cos 60^\circ = \frac{1}{2}$ and $\sin 30^\circ = \frac{1}{2}$. $\sec 60^\circ = \dfrac{1}{\cos 60^\circ} = \dfrac{1}{1/2} = 2$.

$\operatorname{cosec} 30^\circ = \dfrac{1}{\sin 30^\circ} = 2$. Both values are equal to 2.

b. 150° is in the second quadrant, where tangent is negative. $\cot 45^\circ = \dfrac{1}{\tan 45^\circ} = 1$.

$\tan 150^\circ = -\dfrac{1}{\sqrt{3}}$ so $\cot 150^\circ = -\sqrt{3}$. So $\cot 45^\circ = 1$ and $\cot 150^\circ = -\sqrt{3}$.

c. 120° is in the second quadrant (cosine negative) and 240° in the third (sine negative). $\cos 120^\circ = -\frac{1}{2}$, so $\sec 120^\circ = -2$. $\sin 240^\circ = -\dfrac{\sqrt{3}}{2}$, so

$\operatorname{cosec} 240^\circ = -\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$. Rationalising the denominator gives the neat form $-\dfrac{2\sqrt{3}}{3}$.

Answer to Question 3

a. $\cos x = -\frac{1}{2}$ so $x = \mathbf{120°, 240°}$.

b. Use $\tan^2 x = \sec^2 x - 1$: $\sec^2 x - 1 + \sec x = 1$, so $\sec^2 x + \sec x - 2 = 0$,

$(\sec x + 2)(\sec x - 1) = 0$. $\sec x = 1$ gives $x = 0°$. $\sec x = -2$ gives $x = 120°, 240°$. Solutions: **0°, 120°, 240°**.

c. Use $\operatorname{cosec}^2 x = 1 + \cot^2 x$: $\cot^2 x - \cot x - 2 = 0$, $(\cot x - 2)(\cot x + 1) = 0$. $\cot x = 2$: $\tan x = \frac{1}{2}$, $x = 26.6°, 206.6°$. $\cot x = -1$: $\tan x = -1$, $x = 135°, 315°$. Solutions: **26.6°, 135°, 206.6°, 315°**.

Answer to Question 4

Method: use the compound angle formulae $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$, $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ and $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$, with exact values of $30°$, $45°$ and $60°$.

a. Use $\sin(A - B)$ with $A = 45°$ and $B = 30°$.

$\sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30° = \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$

. So $\sin 15° = \dfrac{\sqrt{6} - \sqrt{2}}{4} \approx 0.2588$.

b. Write $75° = 45° + 30°$ and use $\cos(A + B)$.

$\cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30° = \dfrac{\sqrt{6}}{4} - \dfrac{\sqrt{2}}{4} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$. This equals $\sin 15°$, as expected since $\cos 75° = \sin(90° - 75°)$.

c. Use $\tan 60° = \sqrt{3}$ and $\tan 45° = 1$, then rationalise the denominator.

$\tan(60° + 45°) = \dfrac{\sqrt{3} + 1}{1 - \sqrt{3}}$. Multiply by $\dfrac{1 + \sqrt{3}}{1 + \sqrt{3}}$:

$\dfrac{(\sqrt{3} + 1)^2}{1 - 3} = \dfrac{4 + 2\sqrt{3}}{-2} = -(2 + \sqrt{3})$. The answer is negative because $105°$ is in the second quadrant.

Answer to Question 5

a. $2 \sin x \cos x - \sin x = 0$, so $\sin x(2 \cos x - 1) = 0$ (do not divide by $\sin x$). $\sin x = 0$ gives $x = 0°, 180°$. $\cos x = \frac{1}{2}$ gives $x = 60°, 300°$. Solutions: **0°, 60°, 180°, 300°**.

b. Use $\cos 2x = 2\cos^2 x - 1$: $2\cos^2 x + 3\cos x + 1 = 0$, $(2\cos x + 1)(\cos x + 1) = 0$. $\cos x = -\frac{1}{2}$ gives $x = 120°, 240°$. $\cos x = -1$ gives $x = 180°$. Solutions: **120°, 180°, 240°**.

Answer to Question 6

a. $R\sin(x + \alpha) = R\sin x \cos\alpha + R\cos x \sin\alpha$. Comparing: $R\cos\alpha = 5$, $R\sin\alpha = 12$.

$R = \sqrt{25 + 144} = 13$ and $\tan\alpha = \dfrac{12}{5}$, so $\alpha = 67.38°$. Hence $f(x) = 13\sin(x + 67.38°)$.

b. Maximum 13, minimum -13. The maximum occurs when $x + 67.38° = 90°$, so $x = 22.62°$.

c. $13\sin(x + 67.38°) = 6.5$ gives $\sin(x + 67.38°) = 0.5$. So $x + 67.38° = 30°, 150°, 390°$ (the values $30°$ gives a negative x). $x = 82.62°$ or $x = 322.62°$. Solutions: **82.6°, 322.6°**.

Answer to Question 7

Method: the small-angle approximations only work in radians: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$, $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$. Replace each function, simplify, and the powers of θ usually cancel.

a. 0.08 radians is a small angle, so replace $\tan 0.08$ by 0.08. $\tan\theta \approx \theta$, so $h \approx 400 \times 0.08 =$ **32 m**. So the tower is about 32 m tall.

b. Use the calculator in radian mode. $h = 400\tan 0.08 = 400 \times 0.080171 =$ **32.1 m**. So the true height is about 32.1 m.

c. Divide the difference by the true value and multiply by 100. Error $= \dfrac{32.068 - 32}{32.068} \times 100 = 0.21\%$. So the approximation is very good, with an error of about 0.2%.

d. Compare the approximation with the true value for a large angle. The approximation needs θ close to 0. For $\theta = 1$, $\tan 1 = 1.557$ but the approximation gives 1, an error of over 35%. So small-angle approximations should only be used for angles close to zero.

Answer to Question 8

Method: $\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$. Find the sine, cosine or tangent first (with its sign from the quadrant), then take the reciprocal.

a. Use $\operatorname{cosec} 5° = \dfrac{1}{\sin 5°}$ and $\cot 5° = \dfrac{1}{\tan 5°}$. $L = \dfrac{0.45}{\sin 5°} = 5.16$ m and $d = 0.45\cot 5° = \dfrac{0.45}{\tan 5°} = 5.14$ m. The two lengths are close because the angle is small.

b. A gradient of 1 in 12 means a rise of 1 m for every 12 m across. $\tan\theta = \frac{1}{12}$ gives $\theta = \arctan\dfrac{1}{12} = 4.76°$. The ramp must be at most **4.76°**. So the angle must be at most $4.76°$.

c. A smaller angle gives a longer ramp, so the largest allowed angle gives the shortest ramp. The minimum length is $L = 0.45\operatorname{cosec} 4.76° = 5.42$ m and the run is $d = 0.45 \times 12 = 5.4$ m. The 5° ramp is slightly too steep. So the ramp must be at least 5.42 m long, with a horizontal run of 5.4 m.

Answer to Question 9

a. $R = \sqrt{9+16} = 5$, $\tan\alpha = \frac{4}{3}$, $\alpha = 53.13°$. So $h = 6 + 5\sin(30t + 53.13)°$.

b. Greatest depth $6 + 5 = 11$ m, least depth $6 - 5 = 1$ m.

c. Need $5\sin(30t + 53.13)° \geq 3$, so $\sin(30t + 53.13)° \geq 0.6$. $\sin^{-1} 0.6 = 36.87°$, so the argument lies between $36.87°$ and $143.13°$, or between $396.87°$ and $503.13°$. First interval: $0 \leq 30t \leq 90$, giving $0 \leq t \leq 3$. Second interval: $343.74 \leq 30t \leq 360$ (the limit of the range), giving $11.46 \leq t \leq 12$. So boats can enter from midnight until 03:00, and from about 11:28 onwards (11.46 h = 11 h 27.5 min).

Answer to Question 10

a. The range is largest when $\sin 2\theta$ takes its largest value. The maximum value of $\sin 2\theta$ is 1, when $2\theta = 90°$, so $\theta = 45°$. Maximum range $= \frac{400}{9.8} = 40.8$ m. So the maximum range is about 40.8 m, at an angle of $45°$.

b. Substitute $R = 30$, $u = 20$ and $g = 9.8$, then rearrange. $30 = \frac{400\sin 2\theta}{9.8}$ gives $\sin 2\theta = \frac{294}{400} = 0.735$. This is the required result.

c. Since $0°$, the sine equation has two solutions for 2θ. $2\theta = \sin^{-1} 0.735 = 47.31°$ or $180° - 47.31° = 132.69°$. So $\theta = 23.6°$ (a flat kick) or $66.4°$ (a high kick). The two angles add up to $90°$, a general property of projectiles.

Answer to Question 11

Method: use the compound angle formulae $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$, $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ and $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$, with exact values of $30°$, $45°$ and $60°$.

a. Substitute $\tan A = \frac{1}{3}$ and $\tan B = \frac{1}{2}$ into the formula for $\tan(A + B)$.
$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} = \frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{6}} = \frac{5/6}{5/6} = 1$. This is the required result.

b. Both angles are acute, so $A + B$ lies between $0°$ and $180°$; its tangent is positive, so it is acute. $\tan(A + B) = 1$ with $A + B$ acute (here $18.43° + 26.57°$), so $A + B = 45°$. So the path has turned through $45°$ in total.

c. Since $A + B = 45°$, use the exact value of $\sin 45°$. $\sin 45° = \frac{\sqrt{2}}{2}$. So $\sin(A + B) = \frac{\sqrt{2}}{2}$.

d. Use the formula for $\tan(B - A)$ and multiply the top and bottom by 6.

$\tan(B-A) = \dfrac{\frac{1}{2}-\frac{1}{3}}{1+\frac{1}{6}} = \dfrac{1/6}{7/6} = \dfrac{1}{7}$. So $\tan(B-A) = \frac{1}{7}$.

Answer to Question 12

Method: $\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$. Find the sine, cosine or tangent first (with its sign from the quadrant), then take the reciprocal.

a. Use $\operatorname{cosec} 30^\circ = 2$ and $\cot 30^\circ = \sqrt{3}$. Beam length $= 12\operatorname{cosec} 30^\circ = 12 \times 2 = 24$ m. Distance $= 12\cot 30^\circ = 12\sqrt{3} = 20.8$ m. So the beam is 24 m long and lights a spot about 20.8 m away.

b. Rearrange to find $\cot\theta$, then take the reciprocal to get $\tan\theta$. $12\cot\theta = 20$ gives $\tan\theta = \dfrac{12}{20} = 0.6$, so $\theta = 31.0^\circ$. So the beam must make about 31.0° with the horizontal.

c. Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$: $1 + \cot^2\theta = \operatorname{cosec}^2\theta$. With $\cot\theta = \dfrac{20}{12} = \dfrac{5}{3}$: $\operatorname{cosec}^2\theta = 1 + \dfrac{25}{9} = \dfrac{34}{9}$, so $\operatorname{cosec}\theta = \dfrac{\sqrt{34}}{3}$ and the beam length is $12 \times \dfrac{\sqrt{34}}{3} = 4\sqrt{34} = 23.3$ m. So the beam length is $4\sqrt{34} \approx 23.3$ m. Check: $\sqrt{12^2 + 20^2} = \sqrt{544} = 23.3$ by Pythagoras.

Answer to Question 13

Method: the small-angle approximations only work in radians: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$, $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$. Replace each function, simplify, and the powers of θ usually cancel.

a. Replace $\sin\theta$ and $\cos\theta$ by their approximations. $\sin\theta \approx \theta$ gives $L\sin\theta \approx L\theta$. $1 - \cos\theta \approx \dfrac{\theta^2}{2}$ gives $L(1-\cos\theta) \approx \dfrac{L\theta^2}{2}$. These are the required approximations.

b. Substitute $L = 2.5$ and $\theta = 0.2$. Displacement $\approx 2.5 \times 0.2 = 0.5$ m. Rise $\approx \dfrac{2.5 \times 0.04}{2} = 0.05$ m. So the seat moves about 0.5 m sideways and rises about 5 cm.

c. Use the calculator in radian mode for the exact values. Exact displacement $= 2.5\sin 0.2 = 0.4967$ m (error $\dfrac{0.5 - 0.4967}{0.4967} = 0.67\%$). Exact rise $= 2.5(1-\cos 0.2) = 0.04983$ m (error $\dfrac{0.05 - 0.04983}{0.04983} = 0.33\%$). Both errors are below 1%, so the approximations are good for $\theta = 0.2$.

Answer to Question 14

a. Rearrange the double angle formula $\cos 2\theta = 2\cos^2\theta - 1$ to make $\cos^2\theta$ the subject. $\cos 2\theta = 2\cos^2\theta - 1$, so $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$. Hence $I = \dfrac{I_0}{2}(1+\cos 2\theta)$. This is the required result.

b. Divide by I_0 and take the positive square root since θ is acute. $\cos^2\theta = 0.75$ gives $\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\theta = 30°$. So $\theta = 30°$.

c. Divide by I_0, take the positive square root and use the inverse cosine. $\cos^2\theta = 0.2$ gives $\cos\theta = 0.4472$, so $\theta = 63.4°$. So $\theta \approx 63.4°$.

d. $\cos^2\theta \leq 1$, so $I \leq I_0$; the maximum occurs at $\theta = 0°$.

Answer to Question 15

a. LHS $= \dfrac{1}{\sin 2x} + \dfrac{\cos 2x}{\sin 2x} = \dfrac{1+\cos 2x}{\sin 2x} = \dfrac{2\cos^2 x}{2\sin x\cos x} = \dfrac{\cos x}{\sin x} = \cot x$. ■

b. $\sin 3x = \sin(2x+x) = \sin 2x\cos x + \cos 2x\sin x = 2\sin x\cos^2 x + (1-2\sin^2 x)\sin x$. Replace $\cos^2 x = 1-\sin^2 x$: $2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x$. ■

c. $3\sin x - 4\sin^3 x = \sin x$ gives $2\sin x - 4\sin^3 x = 0$, $2\sin x(1-2\sin^2 x) = 0$. $\sin x = 0$: $x = 0°, 180°$. $\sin^2 x = \frac{1}{2}$ and $\sin x > 0$ in the range: $\sin x = \dfrac{1}{\sqrt{2}}$, $x = 45°, 135°$. Solutions: **0°, 45°, 135°, 180°**.

d. With $x = 15°$: $\cot 15° = \operatorname{cosec} 30° + \cot 30° = 2 + \sqrt{3}$. ■

Answer to Question 16

a. $t = 6$: $14 + 3\sin 90° + 4\cos 90° = 14 + 3 + 0 = 17$. $t = 12$: $14 + 3\sin 180° + 4\cos 180° = 14 + 0 - 4 = 10$. $t = 3$: $14 + 3\sin 45° + 4\cos 45° = 14 + \dfrac{7\sqrt{2}}{2} = 18.95 \approx 19.0$.

b. $R = 5$, $\tan\alpha = \dfrac{4}{3}$, $\alpha = 53.13°$. $T = 14 + 5\sin(15t + 53.13)°$. Maximum 19 °C, minimum 9 °C.

c. Maximum when $15t + 53.13 = 90$, so $t = 2.46$ hours after 11:00, that is 13:28.

d. $14 + 5\sin(15t + 53.13)° \geq 17$ gives $\sin(15t + 53.13)° \geq 0.6$. The argument lies in $[36.87°, 143.13°]$ giving $0 \leq t \leq 6$, and in $[396.87°, 503.13°]$, giving $22.92 \leq t \leq 24$ (within range). Total $= 6 + (24 - 22.92) = 7.08$ hours.

Answer to Question 17

a. $\sec^2 x = 1 + \tan^2 x$, so $2 + 2\tan^2 x + 3\tan x - 5 = 0$, i.e. $2\tan^2 x + 3\tan x - 3 = 0$.

b. $\tan x = \dfrac{-3 \pm \sqrt{9+24}}{4} = \dfrac{-3 \pm \sqrt{33}}{4}$, so $\tan x = 0.6861$ or -2.1861.
$\tan x = 0.6861$: $x = 34.5°$ and $214.5°$. $\tan x = -2.1861$: $x = 180° - 65.4° = 114.6°$ and $294.6°$. Solutions: **34.5°, 114.6°, 214.5°, 294.6°**.

c. $\sec x = \dfrac{1}{\cos x}$ and $|\cos x| \le 1$, so $|\sec x| \ge 1$. In the table the values are all at least 1.

d. The tangent has period $180°$, so every value k is reached twice in $360°$. That gives two solutions for each root, four in total.

Answer to Question 18

a. $\cos A = \sqrt{1 - \frac{64}{289}} = \dfrac{15}{17}$, $\sin B = \sqrt{1 - \frac{9}{25}} = \dfrac{4}{5}$ (both positive as the angles are acute).

b. $\sin(A+B) = \sin A\cos B + \cos A \sin B = \dfrac{8}{17}\cdot\dfrac{3}{5} + \dfrac{15}{17}\cdot\dfrac{4}{5} = \dfrac{24+60}{85} = \dfrac{84}{85}$.

c. $\cos(A+B) = \cos A\cos B - \sin A\sin B = \dfrac{45}{85} - \dfrac{32}{85} = \dfrac{13}{85}$. Check: $\dfrac{84^2 + 13^2}{85^2} = \dfrac{7056+169}{7225} = 1$.

d. $\tan A = \dfrac{8}{15}$ and $\tan B = \dfrac{4}{3}$.
$\tan(A-B) = \dfrac{\frac{8}{15} - \frac{4}{3}}{1 + \frac{8}{15}\cdot\frac{4}{3}} = \dfrac{-\frac{12}{15}}{\frac{77}{45}} = -\dfrac{4}{5}\cdot\dfrac{45}{77} = -\dfrac{36}{77}$.

e. $A = 28.07°$ and $B = 53.13°$, so $A + B = 81.2°$. This is acute, consistent with $\cos(A+B) = \frac{13}{85} > 0$.

Answer to Question 19

a. $R = 5$, $\alpha = 53.13°$, so $3\sin x + 4\cos x = 5\sin(x + 53.13°)$.

b. The denominator $7 + 5\sin(x + 53.13°)$ varies from 2 to 12. So B is greatest when the denominator is smallest: $B_{max} = \dfrac{1}{2}$, and $B_{min} = \dfrac{1}{12}$.

c. The denominator is least when $\sin(x + 53.13°) = -1$, so $x + 53.13 = 270$, giving $x = 216.87°$.

d. $7 + 5\sin(x + 53.13°) = 4$ gives $\sin(x + 53.13°) = -0.6$. So $x + 53.13 = 216.87°$ or $323.13°$, giving $x = 163.74°$ or $x = 270°$. Check $x = 270$: $7 - 3 + 0 = 4$. Solutions: **163.7°, 270°**.

Answer to Question 20

a. $3 \times 36^\circ + 2 \times 36^\circ = 180^\circ$, so $3\theta = 180^\circ - 2\theta$ and $\cos 3\theta = \cos(180^\circ - 2\theta) = -\cos 2\theta$.

b. $4c^3 - 3c = -(2c^2 - 1)$, so $4c^3 + 2c^2 - 3c - 1 = 0$.

c. At $c = -1$: $-4 + 2 + 3 - 1 = 0$, so $(c + 1)$ is a factor. Dividing: $4c^3 + 2c^2 - 3c - 1 = (c + 1)(4c^2 - 2c - 1)$. Since $\cos 36^\circ > 0$, $c \neq -1$ so $4c^2 - 2c - 1 = 0$ and $c = \dfrac{2 \pm \sqrt{4 + 16}}{8} = \dfrac{1 \pm \sqrt{5}}{4}$. The positive root gives $\cos 36^\circ = \dfrac{1 + \sqrt{5}}{4}$.

d. $c^2 = \dfrac{(1 + \sqrt{5})^2}{16} = \dfrac{6 + 2\sqrt{5}}{16}$, so $\cos 72^\circ = 2c^2 - 1 = \dfrac{6 + 2\sqrt{5}}{8} - 1 = \dfrac{2\sqrt{5} - 2}{8} = \dfrac{\sqrt{5} - 1}{4}$. Then $\cos 36^\circ - \cos 72^\circ = \dfrac{1 + \sqrt{5} - \sqrt{5} + 1}{4} = \dfrac{1}{2}$. ■