Test answers - Functions, modulus and partial fractions

Year: Year 13 · Duration: 40 minutes · /20

Answer to Question 1 (4 marks)

Mark scheme: a) 1 mark for each composite function; b) 1 mark for the quadratic, 1 mark for both solutions.

a. For fg, substitute $g(x)$ into f; for gf, substitute $f(x)$ into g and expand the bracket. $fg(x) = 2(x^2 - 3) + 1 = 2x^2 - 5$; $gf(x) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2$. The two composite functions are different.
b. Use $gf(x)$ from part a), rearrange to zero and factorise. $4x^2 + 4x - 2 = 6$ gives $4x^2 + 4x - 8 = 0$, $x^2 + x - 2 = 0$, $(x + 2)(x - 1) = 0$, so $x = -2$ or $x = 1$. Check: $gf(1) = 3^2 - 3 = 6$ and $gf(-2) = (-3)^2 - 3 = 6$.

Answer to Question 2 (4 marks)

Method: for $|u| = v$, solve $u = v$ and $u = -v$, then reject any solution that makes v negative. Mark scheme: 1 mark per case, 1 mark for checking; 1 mark for part b).

a. Case 1: $3x - 2 = x + 4$ gives $x = 3$; RHS $= 7 > 0$, valid. Case 2: $-(3x - 2) = x + 4$ gives $-3x + 2 = x + 4$, so $x = -\frac{1}{2}$; RHS $= 3.5 > 0$, valid. Answer: $x = 3$ or $x = -\frac{1}{2}$.
b. $-5 \leq 2x + 1 \leq 5$ gives $-3 \leq x \leq 2$.

Check (a): $|3 \times 3 - 2| = 7 = 3 + 4$ ✓ and $|-\frac{3}{2} - 2| = 3.5 = -\frac{1}{2} + 4$ ✓.

Answer to Question 3 (4 marks)

Method: write the fraction as a sum of simpler fractions, multiply by the denominator and substitute the roots of each linear factor. An improper fraction must be divided first. Mark scheme: 2 marks for each part.

a. $5x + 3 = A(x + 3) + B(x - 1)$. $x = 1$: $8 = 4A$, $A = 2$. $x = -3$: $-12 = -4B$, $B = 3$. So $\dfrac{2}{x - 1} + \dfrac{3}{x + 3}$. Check at $x = 0$: $-2 + 1 = -1$ and $\frac{3}{-3} = -1$.
b. The degrees of numerator and denominator are equal, so the fraction is improper: write $x^2 = (x^2 - 1) + 1$. Improper: $\dfrac{x^2}{x^2 - 1} = 1 + \dfrac{1}{x^2 - 1}$. Then $\dfrac{1}{(x - 1)(x + 1)} = \dfrac{1/2}{x - 1} - \dfrac{1/2}{x + 1}$. So the answer is $1 + \dfrac{1}{2(x - 1)} - \dfrac{1}{2(x + 1)}$.

Answer to Question 4 (4 marks)

Mark scheme: 1 mark for each of parts a) and b), 2 marks for part c).

a. The minimum value of $|x-2|$ is 0, so the range is $h(x) \geq 1$. The minimum value 1 occurs at $x = 2$.
b. Write $|x-2| = 3$, then solve $x - 2 = 3$ and $x - 2 = -3$. $|x-2| = 3$ gives $x = 5$ or $x = -1$. Both values give $h(x) = 3 + 1 = 4$.
c. For $x \geq 2$ the modulus is not needed, so k is a one-to-one linear function. For $x \geq 2$, $k(x) = x - 2 + 1 = x - 1$. Inverse: $k^{-1}(x) = x + 1$, domain $x \geq 1$. The domain of k^{-1} is the range of k.

Answer to Question 5 (4 marks)

Mark scheme: a) 1 mark for both temperatures; b) 1 mark for isolating the exponential, 1 mark for the time; c) 1 mark.

a. Substitute $t = 0$, then let t become very large so that the exponential tends to 0. At $t = 0$: $T = 200$ °C. As $t \to \infty$: $T \to 25$ °C.
b. Subtract 25, divide by 175 and take natural logarithms. $100 = 25 + 175e^{-0.05t}$ gives $e^{-0.05t} = \dfrac{75}{175} = \dfrac{3}{7}$, so $t = -20 \ln \dfrac{3}{7} = 20 \ln \dfrac{7}{3} =$ **16.9 minutes**. So the oven reaches 100 °C after about 17 minutes.
c. The domain of the inverse is the range of T for $t \geq 0$. The inverse is defined for $25 < x \leq 200$.