Lesson - Functions, modulus and partial fractions

Year: Year 13

This chapter covers functions and mappings, composite and inverse functions, the modulus function and partial fractions, then uses functions to model real situations.

Starter activity

A taxi charges £3 plus £2 per mile. Write the fare as a function f of the distance. Now write the distance as a function of the fare. How do the two functions undo each other? Then think of a calculation where you cannot recover the input from the output, such as squaring a number. Why is that different?

Functions and mappings

Definition: function, domain and range

A **function** f gives exactly one output for each input. The set of allowed inputs is the **domain** and the set of outputs is the **range**. A mapping can be one-to-one, many-to-one (such as $f(x) = x^2$, where $f(2) = f(-2)$) or one-to-many, which is *not* a function. Always state the domain: $f(x) = \sqrt{x - 1},\ x \geq 1$.

Worked example 1

Find the range of $f(x) = x^2 - 4x + 7$, $x \in \mathbb{R}$.

Complete the square: $(x - 2)^2 + 3$, so the range is $f(x) \geq 3$.

Composite functions

Definition: composite function

$fg(x)$ means $f(g(x))$: apply g first, then f. The range of g must fit inside the domain of f for fg to exist.

Worked example 2

$f(x) = 2x + 3$ and $g(x) = x^2$. Find $fg(x)$ and $gf(x)$.

$fg(x) = 2x^2 + 3$ while $gf(x) = (2x + 3)^2$. They are different, so order matters.

Worked example 3

$f(x) = 3x - 2$ **and** $g(x) = \dfrac{1}{x+1}$**,** $x \neq -1$**. Solve** $fg(x) = 1$**.**

$fg(x) = \dfrac{3}{x+1} - 2 = 1$ gives $\dfrac{3}{x+1} = 3$, so $x + 1 = 1$ and $x = 0$, which is in the domain of g.

Worked example 4

$g(x) = x^2$ **for all real** x **and** $f(x) = \ln x$**,** $x > 0$**. Find the domain and range of** fg**.**

fg is only defined for $x \neq 0$, because $g(0) = 0$ is not in the domain of f. The range is all real numbers, since x^2 takes every positive value.

Inverse functions

Definition: inverse function

If f is one-to-one it has an inverse f^{-1} with $ff^{-1}(x) = f^{-1}f(x) = x$. The **domain of** f^{-1} **is the range of** f and the range of f^{-1} is the domain of f. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in $y = x$.

Method: finding an inverse

1. Write $y = f(x)$.
2. Make x the subject.
3. Swap the letters, and state the domain of f^{-1} (the range of f).

Worked example 5

Find the inverse of $f(x) = \dfrac{x+1}{x-2}$**,** $x \neq 2$**.**

$y(x-2) = x + 1$, so $yx - 2y = x + 1$, $x(y-1) = 2y + 1$ and $f^{-1}(x) = \dfrac{2x+1}{x-1}$, $x \neq 1$.

Note: restricting the domain

A many-to-one function such as x^2 only has an inverse once the domain is restricted, for example $x \geq 0$. If $f = f^{-1}$ the function is **self-inverse**, for example $f(x) = \dfrac{3x+2}{x-3}$.

The modulus function

Definition: modulus

$|x| = x$ if $x \geq 0$ and $-x$ if x. Its graph is a V-shape with vertex at the origin. For $y = |f(x)|$, reflect any part below the x-axis in the x-axis. For $y = f(|x|)$, keep the part with $x \geq 0$ and reflect it in the y-axis.

Worked example 6

Sketch $y = |2x - 4|$.

Sketch $y = 2x - 4$ (through $(2, 0)$ and $(0, -4)$) and reflect the part below the x-axis. The result is a V-shape with vertex $(2, 0)$ crossing the y-axis at $(0, 4)$. Label both intercepts and the vertex.

Method: modulus equations and inequalities

1. For $|f(x)| = k$ write $f(x) = k$ or $f(x) = -k$.
2. When the right-hand side contains x, **check** each answer: a modulus cannot equal a negative number.
3. $|x - a|$ means $a - b$; $|x - a| > b$ means x or $x > a + b$.
4. With a modulus on both sides, square both sides.

Worked example 7

Solve $|2x - 3| = 5$ and $|x - 1| = 2x - 5$.

First: $2x - 3 = 5$ or -5, so $x = 4$ or $x = -1$. Second: $x - 1 = 2x - 5$ gives $x = 4$ and $2x - 5 = 3 > 0$, valid. $1 - x = 2x - 5$ gives $x = 2$, but $2x - 5 = -1$, so reject. The only solution is $x = 4$.

Worked example 8

Solve $|x + 1|$.

Square: $(x + 1)^2$, so $8x$ and x.

Watch out

Do not write $|x|$ as x only: it means -3. Always check solutions of $|f(x)| = g(x)$.

Partial fractions

Definition: partial fractions

Splitting a rational function into simpler fractions so that it can be integrated or expanded. A fraction is *proper* if the degree of the numerator is less than the degree

of the denominator.

Rule: forms of partial fractions

- Distinct linear factors: $\frac{A}{x-1}+\frac{B}{x+2}$.
- Repeated factor: $\frac{A}{x+1}+\frac{B}{(x+1)^2}$.
- Irreducible quadratic factor: $\frac{Ax+B}{x^2+1}$.
- Improper fraction: divide first, then split the remainder.

Method: finding A and B

1. Check the degree of the numerator and write the correct form.
2. Multiply through by the denominator.
3. Substitute convenient values of x (the cover-up method) and compare coefficients for the rest.
4. Check with a value such as $x=0$.

Worked example 9

Express $\frac{5x+1}{(x-1)(x+2)}$ in partial fractions.

$5x+1=A(x+2)+B(x-1)$. $x=1$: $6=3A$, so $A=2$. $x=-2$: $-9=-3B$, so $B=3$.

The answer is $\frac{2}{x-1}+\frac{3}{x+2}$.

Worked example 10

Express $\frac{3x+5}{(x+1)^2}$ in partial fractions.

$3x+5=A(x+1)+B$. $x=-1$ gives $B=2$; comparing the coefficient of x gives $A=3$. So $\frac{3}{x+1}+\frac{2}{(x+1)^2}$.

Worked example 11

Express $\frac{x^2+3x+3}{(x+1)(x+2)}$ in partial fractions.

Improper: $x^2+3x+3=(x+1)(x+2)+1$, so the fraction is

$1+\frac{1}{(x+1)(x+2)}=1+\frac{1}{x+1}-\frac{1}{x+2}$.

Modelling with functions

Worked example 12

A cup of tea cools according to $T = 20 + 60e^{-0.1t}$ ($^\circ$C, t in minutes). Find the initial temperature, the long-term temperature and the time to reach 50°C.

At $t = 0$, $T = 80^\circ$C. As $t \to \infty$, $T \to 20^\circ$C (room temperature). $T = 50$ gives $e^{-0.1t} = 0.5$, so $t = 10\ln 2 \approx 6.93$ minutes. The inverse $t = -10\ln\left(\frac{T-20}{60}\right)$ gives the time for any T between 20 and 80. The model is valid only for $t \geq 0$.

Watch out

In $fg(x)$ do g first. Do not forget to swap the domain and range of the inverse. Do not divide when the fraction is improper.

Going further with domain and range

Method: finding the range of a function

1. Write down the domain, or the most generous domain that makes the formula work.
2. Sketch the graph if you can; read the lowest and highest y-values.
3. For a quadratic, complete the square. For a rational function, rewrite as a constant plus a fraction.
4. State the range with the correct inequality sign, for example $f(x) \geq 3$ or $f(x) \neq 1$.

Worked example 13: range of a rational function

Find the range of $f(x) = \dfrac{2x+1}{x-1}$, $x \neq 1$.

Rewrite: $f(x) = \dfrac{2(x-1)+3}{x-1} = 2 + \dfrac{3}{x-1}$. The fraction $\dfrac{3}{x-1}$ can be any real number except 0, so $f(x)$ takes every value except 2. Range: $f(x) \in \mathbb{R}$, $f(x) \neq 2$. The same horizontal asymptote $y = 2$ shows on the graph.

Worked example 14: range of an exponential model

Find the range of $f(x) = 5 - 3e^{-x}$, $x \geq 0$.

At $x = 0$, $f = 5 - 3 = 2$. As x increases, e^{-x} decreases towards 0, so f increases towards 5 but never reaches it. Range: $2 \leq f(x)$.

Composite and inverse functions: extra practice

Worked example 15: domain of a composite

$f(x) = \sqrt{x}$, $x \geq 0$ and $g(x) = x - 4$, $x \in \mathbb{R}$. Find $fg(x)$ and state its domain.

$fg(x) = \sqrt{x-4}$. The square root needs $x - 4 \geq 0$, so the domain is $x \geq 4$ (not all of $\mathbb{R}$, even though g is defined everywhere). Always check that the output of the inner function lies in the domain of the outer one.

Worked example 16: inverse with a logarithm

Find the inverse of $f(x) = 3\ln(2x-1) + 4$, $x > \frac{1}{2}$, and state the domain and range of f^{-1}.

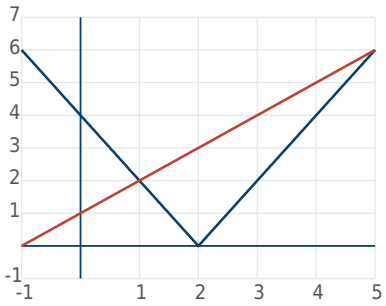
Let $y = 3\ln(2x-1) + 4$. Then $\ln(2x-1) = \dfrac{y-4}{3}$, so $2x - 1 = e^{(y-4)/3}$ and $x = \dfrac{1}{2}\left(e^{(y-4)/3} + 1\right)$. So $f^{-1}(x) = \dfrac{1}{2}\left(e^{(x-4)/3} + 1\right)$. The range of f is all real numbers, so f^{-1} has domain $x \in \mathbb{R}$ and range $f^{-1}(x) > \frac{1}{2}$.

Worked example 17: testing for a self-inverse

Show that $f(x) = \dfrac{4}{x}$, $x \neq 0$, is self-inverse.

$ff(x) = \dfrac{4}{4/x} = x$. Since $ff(x) = x$, $f = f^{-1}$. The graph $y = \dfrac{4}{x}$ is symmetrical in the line $y = x$, which is exactly the geometric meaning of a self-inverse function.

The modulus function: graphs and equations



Note: reading the graph

The blue V-shape is $y = |2x - 4|$ with vertex $(2, 0)$ and y-intercept $(0, 4)$. The red line $y = x + 1$ meets it at two points; the solutions of $|2x - 4| = x + 1$ are the x-coordinates of those points. This sketch is a quick way to see how many solutions to expect and check that the algebra is sensible.

Worked example 18: modulus equation solved by sketching

Solve $|2x-4| = x+1$.

Case 1: $2x-4 = x+1$ gives $x = 5$ (RHS $= 6 > 0$). Case 2: $4-2x = x+1$ gives $x = 1$ (RHS $= 2 > 0$). Both lie on the graph above, so the solutions are $x = 1$ and $x = 5$.

Worked example 19: sketch of the modulus of a quadratic

Sketch $y = |x^2-4|$ and solve $|x^2-4| = 5$.

Sketch $y = x^2 - 4$ (roots ± 2, minimum -4 at $(0,-4)$) and reflect the part below the x-axis. The reflected hump has maximum $(0,4)$. To solve $|x^2-4| = 5$: $x^2-4 = 5$ gives $x = \pm 3$; $x^2-4 = -5$ gives $x^2 = -1$, no real solutions. So $x = \pm 3$. The line $y = 5$ is above the hump of height 4, so it only meets the outer branches.

Partial fractions with an irreducible quadratic factor

Worked example 20: an irreducible quadratic

Express $\dfrac{3x^2+5x+7}{(x+1)(x^2+4)}$ in partial fractions.

The factor x^2+4 has no real roots, so use $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+4}$. Then $3x^2+5x+7 = A(x^2+4) + (Bx+C)(x+1)$. Put $x = -1$: $5 = 5A$, so $A = 1$. Compare x^2: $3 = A + B$, so $B = 2$. Compare constants: $7 = 4A + C$, so $C = 3$. Answer: $\dfrac{1}{x+1} + \dfrac{2x+3}{x^2+4}$. Check at $x = 0$: $\dfrac{7}{4} = 1 + \dfrac{3}{4}$.

Functions as models

Note: what an inverse means in context

When a function models a real situation, its inverse answers the reverse question. If $T(t)$ gives the temperature after t minutes, then $T^{-1}(x)$ gives the time needed to reach a temperature x. The domain of the model is restricted by the context (time cannot be negative), and this restriction decides the range, and hence the domain of the inverse. Always finish a modelling question with a sentence in context, for example "the drink reaches 50 degrees after about 11 minutes".

Worked example 21: a delivery charge

A courier charges $C(d) = 4 + 0.5d$ pounds for a delivery of d kilometres. VAT of 20% is then added, giving $V(c) = 1.2c$. Find $VC(d)$ and the distance for a total of £12.

$VC(d) = 1.2(4 + 0.5d) = 4.8 + 0.6d$. Setting $4.8 + 0.6d = 12$ gives $0.6d = 7.2$ and $d = 12$ km. The inverse of the combined function is $d = \dfrac{y - 4.8}{0.6}$ for $y \geq 4.8$.

Watch out: common errors

- Writing fg when you mean gf. Work from the inside out and show the substitution.
- Giving an inverse without a domain: the domain of f^{-1} is the range of f.
- Forgetting to check solutions of $|f(x)| = g(x)$: reject any root for which $g(x)$.
- Using $A/(x+1) + B/(x+1)$ for a repeated factor instead of $A/(x+1) + B/(x+1)^2$.

Did you know?

The word function was introduced by Gottfried Leibniz in the 1690s and the modern notation $f(x)$ comes from Leonhard Euler in the 1730s. The modulus sign $|x|$ was first used by Karl Weierstrass in 1841. Today functions are everywhere: a spreadsheet formula, a phone app that converts currencies, or a computer program are all examples.

Otto's tip

To check a partial fractions result, substitute one easy value of x (say $x = 0$) into both sides.