Worksheet - Functions, modulus and partial fractions

Year: Year 13 · 20

Show your working.

Question 1

Question 1 – Domain and range *(Section A · Skills practice)*

Find the range of each function. State clearly any restriction on the domain that you use.

a. $f(x) = x^2 - 6x + 10$, $x \in \mathbb{R}$ [3 marks]
b. $g(x) = \sqrt{2x - 4}$, with the largest possible real domain [3 marks]
c. $h(x) = \dfrac{3}{x - 2}$, $x \neq 2$ [2 marks]

Question 2

Question 2 – Composite functions *(Section A · Skills practice)*

The functions f and g are defined for all real x by $f(x) = 3x - 1$ and $g(x) = x^2 + 2$.

a. Find $fg(x)$ and $gf(x)$. [3 marks]
b. Solve $fg(x) = gf(x)$, giving your answers in exact form. [4 marks]
c. Find the value of $ff(2)$. [1 mark]

Question 3

Question 3 – Inverse functions *(Section A · Skills practice)*

Find the inverse function, stating its domain, for each of the following.

a. $f(x) = \dfrac{2x + 1}{x - 3}$, $x \neq 3$ [4 marks]
b. $g(x) = e^{2x} - 1$, $x \in \mathbb{R}$ [3 marks]
c. $h(x) = x^2 + 4$, $x \geq 0$ [3 marks]

Question 4

Question 4 – Modulus equations *(Section A · Skills practice)*

Solve each equation.

a. $|3x - 1| = 8$ [3 marks]

b. $|x+2| = 3x - 2$ [4 marks]
c. $|2x-1| = |x+4|$ [4 marks]

Question 5

Question 5 – Modulus inequalities *(Section A · Skills practice)*

Solve each inequality.

a. $|2x-5|$ [3 marks]
b. $|x+3| \geq 4$ [3 marks]
c. $|x-1|$ [5 marks]

Question 6

Question 6 – Partial fractions *(Section A · Skills practice)*

Express each of the following in partial fractions.

a. $\dfrac{7x+1}{(x+1)(x-2)}$ [3 marks]
b. $\dfrac{4x-1}{(x-1)^2}$ [3 marks]
c. $\dfrac{x^2+1}{x(x+1)}$ [5 marks]

Question 7

Question 7 – The cooling coffee *(Section B · Real-life practice)*

A café in Manchester serves coffee at 90 °C. The temperature in °C after t minutes is modelled by $T(t) = 22 + 68e^{-0.08t}$, $t \geq 0$.

a. State the room temperature suggested by the model and the range of T. [2 marks]
b. Find the inverse function T^{-1} and state its domain. [3 marks]
c. Find, to 3 significant figures, when the coffee has cooled to 60 °C. [2 marks]
d. A customer likes to drink at 50 °C. Find the time from serving. [2 marks]

Question 8

Question 8 – Changing money *(Section B · Real-life practice)*

A traveller changes pounds into euros. The bureau takes a 2% fee first, so the amount after the fee is $f(x) = 0.98x$. The remaining pounds are converted at €1.17 per £, so $g(y) = 1.17y$.

a. Find $gf(x)$ and explain what it represents. [2 marks]

b. How many euros does £250 give? [1 mark]
c. Find the inverse function $(gf)^{-1}$ and use it to find how many pounds must be changed to receive €500, to the nearest penny. [3 marks]
d. Explain why the order of the two operations matters when the fee is a flat amount. [1 mark]

Question 9

Question 9 – Tolerance on bags of flour *(Section B · Real-life practice)*

A factory fills bags of flour. A bag is accepted if its mass m grams satisfies $|m - 500| \leq 8$. The quality-control team weighed six bags.

Bag	A	B	C	D	E	F
Mass (g)	491.5	496.0	508.5	505.0	499.2	507.8

a. Write the acceptance condition without a modulus sign. [1 mark]
b. Which bags are accepted? [2 marks]
c. The scale reads 2.5 g too high. State the accepted range of true masses if the reading must satisfy the original condition. [3 marks]
d. Sketch the graph of $y = |m - 500|$ and mark the region where bags are accepted. [2 marks]

Question 10

Question 10 – Two cyclists on a path *(Section B · Real-life practice)*

Ana and Ben cycle towards each other along a straight 30 m practice track in a park. At time t seconds, Ana is $2t$ metres from the west end and Ben is $30 - 3t$ metres from the west end, for $0 \leq t \leq 10$.

a. Show that the distance between them is $d = |30 - 5t|$. [2 marks]
b. When are they exactly 10 m apart? [3 marks]
c. For how long are they within 5 m of each other? [3 marks]
d. When do they pass each other? [1 mark]

Question 11

Question 11 – The camera lens *(Section B · Real-life practice)*

In photography, an object at distance u cm from a lens of focal length 10 cm forms a sharp image at distance v cm, where $\frac{1}{u} + \frac{1}{v} = \frac{1}{10}$ and $u > 10$.

a. Show that $v = g(u) = \frac{10u}{u - 10}$. [2 marks]
b. Find the image distance for $u = 15$, 20 and 30. [2 marks]
c. Show that $gg(u) = u$, and explain what this means for the inverse function. [3

marks]

d. State the range of g. [2 marks]

Question 12

Question 12 – Saving every year *(Section B · Real-life practice)*

Maya pays £200 into a savings account at the end of each year. The account adds 3% interest during the year. If the balance at the start of a year is £x, the balance at the end is $f(x) = 1.03x + 200$.

a. Find $ff(x)$ and interpret it. [3 marks]
b. Find $f^{-1}(y)$. [2 marks]
c. The balance after two years is £5000. Find the balance at the start of year 2, and at the start of year 1. [3 marks]

Question 13

Question 13 – Printing shirts *(Section B · Real-life practice)*

A school club orders n printed T-shirts. There is a set-up charge of £500 and each T-shirt costs £20 to print. The average cost per T-shirt is $C(n) = \dfrac{500 + 20n}{n}$, $n \geq 1$.

a. Show that $C(n) = 20 + \dfrac{500}{n}$ and find $C(100)$. [2 marks]
b. The club wants an average cost of £22. How many T-shirts are needed? [2 marks]
c. Find the inverse function C^{-1} and state its domain. [3 marks]
d. Explain why the average cost can never fall below £20. [1 mark]

Question 14

Question 14 – Two pumps filling a pool *(Section B · Real-life practice)*

Two pumps fill a leisure-centre pool. Working together they fill it in 4 hours. The slower pump alone would take 6 hours longer than the faster one. Let the faster pump take a hours alone.

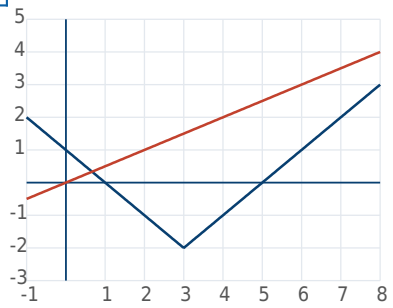
a. Show that $\dfrac{1}{a} + \dfrac{1}{a+6} = \dfrac{1}{4}$ and that this simplifies to $a^2 - 2a - 24 = 0$. [3 marks]
b. Solve to find how long each pump takes alone. [2 marks]
c. Show that $\dfrac{6}{a(a+6)} = \dfrac{1}{a} - \dfrac{1}{a+6}$. [2 marks]
d. Find the fraction of the pool filled by both pumps together in 1 hour and check that it equals the answer from part (a). [2 marks]

Question 15

Question 15 – Sketching a modulus function *(Section C · Problem solving)*

The function f is defined by $f(x) = |x - 3| - 2$, $x \in \mathbb{R}$.

x	0	1	2	3	4	5	6
$f(x)$	1	0	-1	-2	-1	0	1



a. Sketch the graph of $y = f(x)$, labelling the vertex and all intercepts. [3 marks]
b. State the range of f. [1 mark]
c. Solve the equation $f(x) = \frac{x}{2}$. [4 marks]
d. The function g is $f(x)$ restricted to $x \geq 3$. Find $g^{-1}(x)$ and state its domain. [3 marks]
e. Find the values of x for which $f(x)$. [2 marks]

Question 16

Question 16 – Composition and restricted domains *(Section C · Problem solving)*

The functions f and g are defined by $f(x) = 2x - 3$, $x \in \mathbb{R}$ and $g(x) = x^2 + 1$, $x \geq 0$.

x	0	1	2	3
$f(x)$	-3	-1	1	3
$g(x)$	1	2	5	10

a. Find $fg(x)$ and state its range. [3 marks]
b. Explain why $gf(x)$ only exists for $x \geq \frac{3}{2}$ and write down $gf(x)$. [3 marks]
c. Solve $fg(x) = gf(x)$, giving answers to 3 decimal places. [4 marks]
d. Find $g^{-1}(x)$ and state its domain and range. [3 marks]

Question 17

Question 17 – A partial fractions problem *(Section C · Problem solving)*

$$\(F(x) = \frac{3x^2 + 9x - 3}{(x-1)(x+2)^2}, \qquad x > 1.\)$$

a. Find constants A, B and C such that $F(x) = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$. [5 marks]
b. Check your result by evaluating both forms at $x = 2$. [2 marks]
c. Describe the behaviour of $F(x)$ as $x \to \infty$. [2 marks]
d. Show that $F(x) > 0$ for all $x > 1$. [3 marks]

Question 18

Question 18 – Quality control on piston rings *(Section C · Problem solving)*

A machine cuts piston rings to length L mm. Rings are accepted if $|L - 120| \leq 0.4$. An inspector records the total cost of waste as $5|L - 120|$ pounds per ring, because rings further from 120 mm need more rework.

Ring	1	2	3	4	5	6
L (mm)	119.7	120.5	120.2	119.5	120.4	119.9

a. Find the accepted range of L and decide which rings are accepted. [3 marks]
b. Find the rework cost for each ring. [3 marks]
c. The company will only accept rework costs below £1.50. Solve $5|L - 120|$. [3 marks]
d. The gauge reads 0.15 mm too high. Find the true accepted range of lengths when the readings must satisfy $|r - 120| \leq 0.4$. [3 marks]

Question 19

Question 19 – A population of seals *(Section C · Problem solving)*

The number of grey seals on an island is modelled by $P(t) = \dfrac{800}{1 + 3e^{-0.4t}}$, where t is the time in years after monitoring began.

t (years)	0	2	5	10
$P(t)$	200	341	569	758

a. Verify the value in the table at $t = 5$ and state the long-term population predicted by the model. [3 marks]
b. State the range of P for $t \geq 0$. [2 marks]
c. Find the inverse function $t = P^{-1}(N)$. [4 marks]
d. Use your inverse to find when the population first reaches 600. [2 marks]
e. Comment on whether the model is sensible. [2 marks]

Question 20

Question 20 – Self-inverse functions *(Section D · Challenge)*

The function f is defined by $f(x) = \dfrac{3x + k}{x - 3}$, $x \in \mathbb{R}$, $x \neq 3$, where k is a constant.

a. Show that $ff(x) = x$ for every k, and explain what this means. [4 marks]
b. Hence state $f^{-1}(x)$. [1 mark]
c. Show that $f(x) = x$ leads to $x^2 - 6x - k = 0$. [2 marks]
d. Find the range of values of k for which $f(x) = x$ has two distinct real roots. [2 marks]
e. For $k = 7$ find the exact solutions of $f(x) = f^{-1}(x)$. [2 marks]