Answers - Functions, modulus and partial fractions

Year: Year 13

Answer to Question 1

a. Complete the square: $f(x) = (x-3)^2 + 1$. A square is at least 0, so the range is $f(x) \geq 1$.
b. We need $2x - 4 \geq 0$, so the domain is $x \geq 2$. At $x = 2$, $g = 0$ and g increases without limit. Range: $g(x) \geq 0$.
c. The fraction $\frac{3}{x-2}$ takes every value except 0 (a numerator of 3 can never produce 0). Range: $h(x) \in \mathbb{R}$, $h(x) \neq 0$.

Answer to Question 2

Method: $fg(x)$ means apply g first, then f. Replace every x in the outer function by the whole inner expression.

a. For fg, substitute $x^2 + 2$ into f; for gf, substitute $3x - 1$ into g. $fg(x) = 3(x^2+2) - 1 = 3x^2 + 5$. $gf(x) = (3x-1)^2 + 2 = 9x^2 - 6x + 3$. The two results are different: composition is not commutative.
b. Set the two expressions equal and collect everything on one side. The quadratic does not factorise, so use the formula. $3x^2 + 5 = 9x^2 - 6x + 3$ gives $6x^2 - 6x - 2 = 0$, i.e. $3x^2 - 3x - 1 = 0$. By the formula $x = \frac{3 \pm \sqrt{9+12}}{6} = \frac{3 \pm \sqrt{21}}{6}$. So $x = \frac{3+\sqrt{21}}{6}$ or $x = \frac{3-\sqrt{21}}{6}$.
c. Work from the inside: first find $f(2)$, then apply f to the result. $f(2) = 5$ and $f(5) = 14$, so $ff(2) = 14$. So $ff(2) = 14$.

Answer to Question 3

a. Let $y = \frac{2x+1}{x-3}$. Then $y(x-3) = 2x + 1$, $xy - 3y = 2x + 1$, $x(y-2) = 3y + 1$, so $x = \frac{3y+1}{y-2}$. Hence $f^{-1}(x) = \frac{3x+1}{x-2}$, $x \neq 2$ (the range of f excludes 2).
b. $y = e^{2x} - 1$ gives $e^{2x} = y + 1$, $2x = \ln(y+1)$. So $g^{-1}(x) = \frac{1}{2}\ln(x+1)$, domain $x > -1$ (the range of g).
c. $y = x^2 + 4$ with $x \geq 0$ gives $x = \sqrt{y-4}$. So $h^{-1}(x) = \sqrt{x-4}$, domain $x \geq 4$.

Answer to Question 4

a. $3x - 1 = 8$ gives $x = 3$. $3x - 1 = -8$ gives $x = -\frac{7}{3}$.

b. Case 1: $x + 2 = 3x - 2$ gives $x = 2$. Check: $3(2) - 2 = 4 > 0$ and $|2 + 2| = 4$. Case 2: $-(x + 2) = 3x - 2$ gives $-x - 2 = 3x - 2$, so $x = 0$. Check: $3(0) - 2 = -2$, impossible since a modulus cannot be negative, so reject. Only $x = 2$.

c. Square both sides: $(2x - 1)^2 = (x + 4)^2$, so $4x^2 - 4x + 1 = x^2 + 8x + 16$, $3x^2 - 12x - 15 = 0$, $x^2 - 4x - 5 = 0$, $(x - 5)(x + 1) = 0$. So $x = 5$ or $x = -1$. Check: $|9| = |9|$ and $|-3| = |3|$.

Answer to Question 5

a. -7 gives -2, so -1.

b. $x + 3 \geq 4$ or $x + 3 \leq -4$, so $x \geq 1$ or $x \leq -7$.

c. Both sides must be considered: the right-hand side $2x$ must be positive, so $x > 0$. Squaring: $(x - 1)^2$, so $3x^2 + 2x - 1 > 0$, $(3x - 1)(x + 1) > 0$, giving x or $x > \frac{1}{3}$. Combined with $x > 0$: $x > \frac{1}{3}$. Check $x = 1$: $|0|$ true; $x = 0.2$: 0.8 false.

Answer to Question 6

a. Write $\frac{A}{x+1} + \frac{B}{x-2}$, so $7x + 1 = A(x - 2) + B(x + 1)$. $x = -1$: $-6 = -3A$, $A = 2$. $x = 2$: $15 = 3B$, $B = 5$. Answer: $\frac{2}{x+1} + \frac{5}{x-2}$.

b. Write $\frac{A}{x-1} + \frac{B}{(x-1)^2}$, so $4x - 1 = A(x - 1) + B$. $x = 1$: $B = 3$. Coefficient of x: $A = 4$. Answer: $\frac{4}{x-1} + \frac{3}{(x-1)^2}$.

c. Numerator and denominator both have degree 2, so divide first: $x^2 + 1 = 1 \cdot (x^2 + x) + (1 - x)$. So the fraction is $1 + \frac{1-x}{x(x+1)}$. Then $\frac{1-x}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$ with $1 - x = A(x + 1) + Bx$: $x = 0$: $A = 1$; $x = -1$: $2 = -B$, $B = -2$. Answer: $1 + \frac{1}{x} - \frac{2}{x+1}$.

Answer to Question 7

Method: to find an inverse, write $y = f(x)$, make x the subject, then swap the letters. The domain of f^{-1} is the range of f.

a. Look at what happens to the exponential term as t becomes very large. As $t \to \infty$, $e^{-0.08t} \to 0$ so $T \to 22$ °C (room temperature). At $t = 0$, $T = 90$. Range: 22. So

the coffee never quite reaches room temperature.

b. Subtract 22, divide by 68 and take natural logarithms. $y = 22 + 68e^{-0.08t}$ gives $e^{-0.08t} = \dfrac{y-22}{68}$, so $t = -\dfrac{1}{0.08}\ln\dfrac{y-22}{68} = -12.5\ln\dfrac{y-22}{68}$. So $T^{-1}(x) = -12.5\ln\left(\dfrac{x-22}{68}\right)$ with domain 22. The domain of T^{-1} is the range of T found in part a).

c. Substitute 60 into the inverse function. $t = -12.5\ln\dfrac{38}{68} = 12.5 \times 0.5817 =$ **7.27** minutes. So the coffee reaches 60 °C after about 7.27 minutes.

d. Substitute 50 into the inverse function. $t = -12.5\ln\dfrac{28}{68} = 12.5 \times 0.8873 =$ **11.1** minutes. So the customer should wait about 11.1 minutes.

Answer to Question 8

a. $gf(x) = 1.17 \times 0.98x = 1.1466x$. It converts £$x$ to euros after the fee.

b. $1.1466 \times 250 =$ **€286.65**.

c. Let $y = 1.1466x$, so $(gf)^{-1}(y) = \dfrac{y}{1.1466}$. For €500: $\dfrac{500}{1.1466} =$ **£436.07**.

d. With a percentage fee the order does not change the final answer, but a flat fee of, say, £3 must be taken from the pounds before conversion, since £3 taken from euros would be worth less. The order of composition is a modelling decision.

Answer to Question 9

a. $-8 \le m - 500 \le 8$, so $492 \le m \le 508$.

b. A (491.5) is below 492 and C (508.5) is above 508, so both are rejected. B (496.0), D (505.0), E (499.2) and F (507.8) all lie inside the range. **Accepted: B, D, E, F.**

c. Reading $r = m + 2.5$ satisfies $|r - 500| \le 8$, so $492 \le m + 2.5 \le 508$ and $489.5 \le m \le 505.5$.

d. A V-shape with vertex $(500, 0)$. The accepted region is the part of the graph below the line $y = 8$, between $m = 492$ and $m = 508$.

Answer to Question 10

Method: a distance between two positions is always the modulus of their difference. An equation $|u| = k$ gives $u = k$ or $u = -k$; an inequality $|u|$ gives $-k$.

a. Subtract the two positions and take the modulus, since a distance cannot be negative. $d = |(30 - 3t) - 2t| = |30 - 5t|$. This is the required result.

b. Solve $|30 - 5t| = 10$ by considering both signs. $30 - 5t = 10$ gives $t = 4$; $30 - 5t = -10$ gives $t = 8$. So at **4 s** and **8 s**. At 4 s they have not yet met; at 8 s

they have already passed each other.

c. Solve the double inequality. $|30 - 5t|$ gives -5, so -35, i.e. 5. They are within 5 m for **2 s**. So they are within 5 m of each other between 5 s and 7 s.

d. They pass when the distance between them is zero. When $d = 0$: $t = 6$ s. So they pass each other after 6 seconds, 12 m from the west end.

Answer to Question 11

Method: to find an inverse, write $y = f(x)$, make x the subject, then swap the letters. The domain of f^{-1} is the range of f.

a. Rearrange the lens equation to make $\frac{1}{v}$ the subject, then take the reciprocal. $\frac{1}{v} = \frac{1}{10} - \frac{1}{u} = \frac{u - 10}{10u}$, so $v = \frac{10u}{u - 10}$. This is the required formula.

b. Substitute each value of u. $g(15) = \frac{150}{5} = 30$; $g(20) = \frac{200}{10} = 20$; $g(30) = \frac{300}{20} = 15$. The image distances are 30 cm, 20 cm and 15 cm.

c. Substitute $g(u)$ into g and multiply the top and bottom by $u - 10$. $gg(u) = \frac{10 \cdot \frac{10u}{u-10}}{\frac{10u}{u-10} - 10} = \frac{100u}{10u - 10(u - 10)} = \frac{100u}{100} = u$. So g is **self-inverse**: $g^{-1} = g$. Swapping object and image distances gives the same lens equation. Part b) illustrates this: $g(15) = 30$ and $g(30) = 15$.

d. Write $g(u)$ as a constant plus a fraction. $g(u) = 10 + \frac{100}{u - 10}$ with $u > 10$, so $g(u) > 10$. Range: $g(u) > 10$. So the image is always more than 10 cm from the lens.

Answer to Question 12

Method: to find an inverse, write $y = f(x)$, make x the subject, then swap the letters. The domain of f^{-1} is the range of f.

a. Apply f to the expression $1.03x + 200$.
$ff(x) = 1.03(1.03x + 200) + 200 = 1.0609x + 406$. It gives the balance after two years. So $ff(x) = 1.0609x + 406$.

b. Subtract 200 and divide by 1.03. $y = 1.03x + 200$ gives $x = \frac{y - 200}{1.03}$. So $f^{-1}(y) = \frac{y - 200}{1.03}$. The inverse function undoes one year of saving.

c. Apply the inverse function twice, working backwards one year at a time. Start of year 2: $f^{-1}(5000) = \frac{4800}{1.03} =$ **£4660.19**. Start of year 1: $f^{-1}(4660.19) = \frac{4460.19}{1.03} =$ **£4330.29**. Check: $\frac{5000 - 406}{1.0609} = 4330.29$. So Maya

started with £4330.29.

Answer to Question 13

Method: to find an inverse, write $y = f(x)$, make x the subject, then swap the letters. The domain of f^{-1} is the range of f.

a. Divide each term of the numerator by n. Splitting the fraction: $\frac{500}{n} + \frac{20n}{n} = \frac{500}{n} + 20$. $C(100) = 20 + 5 =$ **£25**. So 100 T-shirts cost £25 each on average.

b. Set $C(n) = 22$ and solve for n. $20 + \frac{500}{n} = 22$ gives $\frac{500}{n} = 2$, $n =$ **250**. So the club needs 250 T-shirts.

c. Make n the subject of $y = 20 + \frac{500}{n}$. $y = 20 + \frac{500}{n}$ gives $n = \frac{500}{y - 20}$. So $C^{-1}(x) = \frac{500}{x - 20}$, domain 20 (since $C(1) = 520$). The domain of the inverse is the range of C.

d. Look at the sign of the fraction. $\frac{500}{n} > 0$ for every n, so $C(n) > 20$; the average approaches the print cost £20 as n grows. So the set-up charge is spread over more shirts but never disappears.

Answer to Question 14

a. In one hour the fast pump fills $\frac{1}{a}$ of the pool and the slow pump $\frac{1}{a+6}$, and together $\frac{1}{4}$. Multiply by $4a(a+6)$: $4(a+6) + 4a = a(a+6)$, so $8a + 24 = a^2 + 6a$, $a^2 - 2a - 24 = 0$.

b. $(a-6)(a+4) = 0$, so $a = 6$ (a time cannot be -4). The faster pump takes **6 h** and the slower **12 h**.

c. Put over a common denominator: $\frac{(a+6) - a}{a(a+6)} = \frac{6}{a(a+6)}$.

d. $\frac{1}{6} + \frac{1}{12} = \frac{2+1}{12} = \frac{1}{4}$. They fill $\frac{1}{4}$ of the pool per hour, so 4 hours in total.

Answer to Question 15

a. The graph is a V-shape with vertex $(3, -2)$. y-intercept: $f(0) = 3 - 2 = 1$, so $(0, 1)$. x-intercepts: $|x - 3| = 2$, so $x = 1$ and $x = 5$: $(1, 0)$ and $(5, 0)$.

b. The minimum value is -2: range $f(x) \geq -2$.

c. $|x - 3| = \frac{x}{2} + 2$. Case 1: $x - 3 = \frac{x}{2} + 2$ gives $\frac{x}{2} = 5$, $x = 10$; check RHS $= 7 > 0$.

Case 2: $3 - x = \frac{x}{2} + 2$ gives $1 = \frac{3x}{2}$, $x = \frac{2}{3}$; check RHS $= \frac{7}{3} > 0$. So $x = 10$ or $x = \frac{2}{3}$.

d. For $x \geq 3$, $g(x) = x - 3 - 2 = x - 5$. Let $y = x - 5$, so $x = y + 5$: $g^{-1}(x) = x + 5$, domain $x \geq -2$ (the range of g).

e. $|x - 3| - 2$ gives $|x - 3|$, so 0.

Answer to Question 16

a. $fg(x) = 2(x^2 + 1) - 3 = 2x^2 - 1$ with $x \geq 0$. The minimum is -1 at $x = 0$, so the range is $fg(x) \geq -1$.

b. The domain of g is $x \geq 0$, so the input $f(x) = 2x - 3$ must satisfy $2x - 3 \geq 0$, that is $x \geq \frac{3}{2}$. Then $gf(x) = (2x - 3)^2 + 1 = 4x^2 - 12x + 10$.

c. $2x^2 - 1 = 4x^2 - 12x + 10$ gives $2x^2 - 12x + 11 = 0$, so $x = \frac{12 \pm \sqrt{144 - 88}}{4} = \frac{12 \pm \sqrt{56}}{4} = 3 \pm \frac{\sqrt{14}}{2}$, i.e. $x = 4.871$ or 1.129. Only $x \geq \frac{3}{2}$ is allowed, so $x = 4.871$.

d. $y = x^2 + 1$ with $x \geq 0$ gives $x = \sqrt{y - 1}$. $g^{-1}(x) = \sqrt{x - 1}$ with domain $x \geq 1$ and range $g^{-1}(x) \geq 0$.

Answer to Question 17

a. $3x^2 + 9x - 3 = A(x + 2)^2 + B(x - 1)(x + 2) + C(x - 1)$. $x = 1$: $9 = 9A$, $A = 1$. $x = -2$: $12 - 18 - 3 = -9 = -3C$, $C = 3$. Coefficient of x^2: $3 = A + B$, so $B = 2$. Answer: $\frac{1}{x - 1} + \frac{2}{x + 2} + \frac{3}{(x + 2)^2}$.

b. At $x = 2$: original $= \frac{12 + 18 - 3}{(1)(16)} = \frac{27}{16}$. Partial fractions: $1 + \frac{2}{4} + \frac{3}{16} = \frac{16 + 8 + 3}{16} = \frac{27}{16}$. They agree.

c. Each term tends to 0, and the dominant term is $\frac{1}{x} + \frac{2}{x} = \frac{3}{x}$, so $F(x) \to 0$ from above (the graph approaches the x-axis).

d. For $x > 1$: $x - 1 > 0$, $x + 2 > 0$ and $(x + 2)^2 > 0$, so every term is positive and their sum is positive.

Answer to Question 18

a. $119.6 \leq L \leq 120.4$. Ring 1 (0.3 away) accepted, ring 2 (0.5) rejected, ring 3 (0.2) accepted, ring 4 (0.5) rejected, ring 5 (0.4) accepted, ring 6 (0.1) accepted.

b. Cost $= 5\times$ distance: ring 1 £1.50, ring 2 £2.50, ring 3 £1.00, ring 4 £2.50, ring 5

£2.00, ring 6 £0.50.

c. $|L - 120|$ so 119.7.

d. The reading is $r = L + 0.15$, so $|L + 0.15 - 120| \leq 0.4$ gives $-0.4 \leq L - 119.85 \leq 0.4$, so $119.45 \leq L \leq 120.25$.

Answer to Question 19

a. $e^{-2} = 0.13534$, so $3e^{-2} = 0.40601$ and $P(5) = \dfrac{800}{1.40601} = 569$, which matches the table. As $t \to \infty$, $e^{-0.4t} \to 0$ and $P \to 800$.

b. $P(0) = 200$ and P increases towards 800, so the range is $200 \leq P$.

c. $N = \dfrac{800}{1 + 3e^{-0.4t}}$ gives $1 + 3e^{-0.4t} = \dfrac{800}{N}$, so $e^{-0.4t} = \dfrac{800/N - 1}{3}$, and $t = -2.5 \ln\left(\dfrac{800 - N}{3N}\right)$ with domain $200 \leq N$.

d. $N = 600$: $t = -2.5 \ln \dfrac{200}{1800} = -2.5 \ln \frac{1}{9} = 2.5 \ln 9 =$ **5.49 years**.

e. The model predicts a carrying capacity of 800, which is realistic for limited food and space, and it starts at 200 as observed. It cannot allow for sudden events such as disease, and a real population fluctuates.

Answer to Question 20

a. $ff(x) = \dfrac{3\left(\frac{3x+k}{x-3}\right) + k}{\frac{3x+k}{x-3} - 3} = \dfrac{3(3x+k) + k(x-3)}{(3x+k) - 3(x-3)} = \dfrac{9x + 3k + kx - 3k}{k+9} = \dfrac{x(9+k)}{k+9} = x$ (provided $k \neq -9$). So f is self-inverse: it undoes itself.

b. $f^{-1}(x) = f(x) = \dfrac{3x + k}{x - 3}$.

c. $\dfrac{3x + k}{x - 3} = x$ gives $3x + k = x^2 - 3x$, so $x^2 - 6x - k = 0$.

d. Discriminant $36 + 4k > 0$ gives $k > -9$.

e. For $k = 7$ the equation $f(x) = f^{-1}(x)$ is the same as $f(x) = x$ because $f = f^{-1}$, so $x^2 - 6x - 7 = 0$, $(x - 7)(x + 1) = 0$, so $x = 7$ or $x = -1$.