Lesson - Forces in two dimensions and friction

Year: Year 13

When several forces act on a particle they combine to a single **resultant**. This chapter resolves forces into components, uses equilibrium and friction, and solves problems on slopes and with connected particles.

Starter activity

Place a book on a wooden board and slowly lift one end. At what angle does the book start to slide? Repeat with a heavier book on top. Does the angle change? Sketch the forces on the book and discuss which forces act parallel to the board and which act perpendicular to it. Predict what would happen on an icy surface.

Forces as vectors

Definition: force, weight, tension and normal reaction

A **force** has magnitude (newtons, N) and direction. **Weight** is $W = mg$ acting down ($g = 9.8\ \text{m}\,\text{s}^{-2}$). A **tension** acts along a string away from the particle. A surface exerts a **normal reaction** R perpendicular to itself.

Rule: resolving a force

A force F at angle θ to the horizontal has horizontal component $F\cos\theta$ and vertical component $F\sin\theta$. If the angle is to the vertical the roles swap, so always draw the triangle first. A 20 N force at 30° above the horizontal has components 17.3 N and 10 N.

Finding the resultant

Method: resultant of several forces

1. Resolve every force into horizontal and vertical components.
2. Add the components in each direction.
3. Find the magnitude with Pythagoras and the direction with $\tan^{-1}$.

Worked example 1

Forces of 10 N along the x-axis and 8 N at 60° above it act on a particle. Find the resultant.

Components $10 + 8\cos 60^\circ = 14$ and $8\sin 60^\circ = 6.93$. Magnitude $\sqrt{14^2 + 6.93^2} = 15.6$ N, direction $\tan^{-1}\dfrac{6.93}{14} = 26.3^\circ$ above the x-axis.

Worked example 2

Forces $(4\mathbf{i} + 3\mathbf{j})$ N and $(-\mathbf{i} + 5\mathbf{j})$ N act on a 2 kg particle with a further force so the resultant is $(6\mathbf{i} + 8\mathbf{j})$ N. Find the resultant of the first two and the acceleration.

The first two give $3\mathbf{i} + 8\mathbf{j}$, magnitude $\sqrt{73} = 8.54$ N at 69.4° to $\mathbf{i}$. Under a resultant $(6\mathbf{i} + 8\mathbf{j})$ N, $\mathbf{a} = \dfrac{\mathbf{F}}{m} = (3\mathbf{i} + 4\mathbf{j})\ \mathrm{m\,s^{-2}}$, magnitude $5\ \mathrm{m\,s^{-2}}$.

Equilibrium

Definition: equilibrium

A particle is in **equilibrium** when the resultant force is zero: the components in each of two perpendicular directions sum to zero.

Worked example 3

A 50 N weight hangs from two strings at 30° and 60° to the horizontal. Find the tensions.

Horizontally $T_1\cos 30^\circ = T_2\cos 60^\circ$, so $T_2 = \sqrt{3}\,T_1$. Vertically $T_1\sin 30^\circ + T_2\sin 60^\circ = 50$, giving $0.5T_1 + 1.5T_1 = 50$, so $T_1 = 25$ N and $T_2 = 43.3$ N.

Worked example 4

A 40 N lamp hangs from two equal cables, each at 30° to the horizontal. Find T. Then find $\mathbf{F}_3$ if $\mathbf{F}_1 = (2\mathbf{i} - \mathbf{j})$ and $\mathbf{F}_2 = (-5\mathbf{i} + 4\mathbf{j})$ are in equilibrium with it.

$2T\sin 30^\circ = 40$, so $T = 40$ N: a shallow angle needs a large tension. For vectors set the sum to zero: $\mathbf{F}_3 = (3\mathbf{i} - 3\mathbf{j})$ N.

Friction

Rule: friction

Friction acts parallel to the surface, **opposing the motion or the tendency to move**, and $F \le \mu R$, where μ is the coefficient of friction. At rest F is only as large as needed. In **limiting equilibrium** (about to slip) or when sliding, $F = \mu R$.

Worked example 5

A 20 kg crate on a horizontal floor has $\mu = 0.35$. Find the push needed to move it, and the pull P at 25° above the horizontal that keeps it moving at constant speed.

$R = 196$ N, limiting friction 68.6 N, so a push of 68.6 N is just enough. With the rope: $R = 196 - P\sin 25^\circ$ and $P\cos 25^\circ = 0.35R$, giving $P = 65.1$ N. A rope above the horizontal reduces R and so reduces friction.

Watch out

Do not assume $R = mg$ when another force has a vertical component. Use $F = \mu R$ only in limiting equilibrium or when sliding. Draw friction against the motion.

Inclined planes

Rule: rough slope at angle α

The weight has components $mg\sin\alpha$ down the slope and $mg\cos\alpha$ into it, so $R = mg\cos\alpha$ with no other perpendicular forces. A block stays in equilibrium if $\tan\alpha \le \mu$. Sliding down: $a = g(\sin\alpha - \mu\cos\alpha)$.

Worked example 6

A 4 kg block rests on a rough 30° plane. Find F, R and the least μ.

$F = mg\sin 30^\circ = 19.6$ N and $R = 33.9$ N, so $\mu \ge \dfrac{19.6}{33.9} = \tan 30^\circ = 0.577$.

Worked example 7

(a) On a 25° plane with $\mu = 0.4$ does a block slide? (b) What force parallel to a 20° plane pushes a 6 kg block up at constant speed if $\mu = 0.25$?

(a) $\tan 25^\circ = 0.466 > 0.4$, so it slides, with $a = g(\sin 25^\circ - 0.4\cos 25^\circ) = 0.589\ \mathrm{m\,s^{-2}}$.
(b) $P = mg\sin 20^\circ + \mu mg\cos 20^\circ = 33.9$ N.

Connected particles

Worked example 8

A 3 kg block on a rough table ($\mu = 0.2$) is joined by a string over a smooth pulley to a hanging 2 kg block. Find a and T.

$2g - T = 2a$ and $T - 0.2(3g) = 3a$, so $a = \dfrac{2g - 0.6g}{5} = 2.74\ \mathrm{m\,s^{-2}}$ and $T = 2(g - a) = 14.1$ N. The string is light and inextensible, so tension and acceleration

are the same for both particles.

Watch out

Do not swap sine and cosine when resolving. Check $F \leq \mu R$ for equilibrium on a slope.

Worked example 9: Resolving at an angle to the vertical

A force of 50 N acts at 20° to the vertical. Find its horizontal and vertical components.

The angle is measured from the vertical, so the vertical component is the **adjacent** side and the horizontal component is the **opposite** side. Vertical $= 50\cos 20^\circ = 47.0$ N. Horizontal $= 50\sin 20^\circ = 17.1$ N. Sketch the right-angled triangle first: the angle given is next to the component with cosine.

Worked example 10: A resultant using i and j

Forces of $(5\mathbf{i} - 2\mathbf{j})$ N, $(-3\mathbf{i} + 7\mathbf{j})$ N and $(\mathbf{i} + \mathbf{j})$ N act on a particle of mass 3 kg. Find the acceleration.

Add: $(5 - 3 + 1)\mathbf{i} + (-2 + 7 + 1)\mathbf{j} = 3\mathbf{i} + 6\mathbf{j}$ N. $\mathbf{a} = \frac{1}{3}(3\mathbf{i} + 6\mathbf{j}) = (\mathbf{i} + 2\mathbf{j})$ m s^{-2} with magnitude $\sqrt{5} = 2.24$ m s^{-2}. The direction is $\tan^{-1} 2 = 63.4^\circ$ above $\mathbf{i}$.

Worked example 11: Limiting friction in equilibrium

A 10 kg box rests on a rough floor, $\mu = 0.5$. A horizontal force of 30 N is applied. Does the box move? What is the friction force?

$R = 98$ N so the maximum friction is $0.5 \times 98 = 49$ N. The applied force 30 N is less than 49 N, so the box stays at rest. Friction is only as large as needed: $F = 30$ N, **not** 49 N. The formula $F = \mu R$ applies only when the box is about to move or moving.

Worked example 12: Atwood machine: two hanging masses

Masses of 5 kg and 3 kg hang from the ends of a light string over a smooth pulley. Find the acceleration and the tension.

Heavier mass moves down: $5g - T = 5a$. Lighter mass moves up: $T - 3g = 3a$. Add: $2g = 8a$ so $a = \dfrac{2 \times 9.8}{8} = 2.45$ m s^{-2}. Then $T = 3(g + a) = 3(12.25) = 36.75$ N. Check with the other mass: $T = 5(g - a) = 5(7.35) = 36.75$ N.

Worked example 13: Pushing a block up a smooth slope at constant speed

What force parallel to a smooth 20° slope pushes a 6 kg block up the slope at

constant speed? Then repeat if $\mu = 0.25$.

Smooth: constant speed means zero resultant, so $P = mg\sin 20^\circ = 6 \times 9.8 \times 0.3420 = 20.1$ N. Rough: friction acts down the slope, $F = \mu mg\cos 20^\circ = 0.25 \times 55.26 = 13.8$ N. So $P = 20.1 + 13.8 = 33.9$ N. The rough slope needs 69% more force.

Worked example 14: Finding the angle at which a body slips

A box rests on a rough ramp whose coefficient of friction with the box is 0.6. Find the steepest angle at which the box can rest, and decide whether a ramp at 35° is safe.

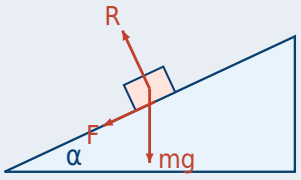
A body rests in equilibrium only while $\tan\alpha \le \mu$, so the steepest angle is $\tan^{-1} 0.6 = 31.0^\circ$. A ramp at 35° has $\tan 35^\circ = 0.700 > 0.6$, so friction cannot hold the box and it will slide. The mass of the box is irrelevant, because both the weight component and the maximum friction are proportional to m.

Note: Newton third law and tension

When a string pulls a block, the block pulls back on the string with an equal and opposite force. In a problem with a light string, the tension has the same magnitude at both ends, which is why a single symbol T can be used on both particles. If the string passes over a rough pulley or has mass, the tensions on either side would differ, but at this level pulleys are assumed smooth and strings light.

Note: what a force diagram looks like

Draw the object as a dot or box and show every force as an arrow pointing the way it acts. On a slope, the weight points straight down but you resolve it parallel and perpendicular to the plane. Friction acts along the plane and opposes motion.



Method: a force problem in six steps

1. Draw a large force diagram and mark all forces, including weight mg, normal reaction R, tension and friction F.
2. Choose two perpendicular directions (parallel and perpendicular to the plane on a slope).
3. Resolve every force into these directions.
4. Write an equation for each direction: sum of forces $= ma$ (or 0 in equilibrium).

5. Use $F = \mu R$ if the particle is moving or about to move.
6. Solve, check that the answer is sensible, and give units.

Watch out: friction and the normal reaction

- $R = mg$ only if no other force has a vertical component. A rope pulling upwards reduces R; a push downwards increases it.
- On a slope $R = mg\cos\alpha$ only if there are no other forces perpendicular to the plane.
- Friction always opposes the motion (or the tendency to move), so it points up the slope for a block sliding down.
- For connected particles, write a separate equation for each and use the same acceleration in both because the string is inextensible.

Did you know?

The unit of force is named after Isaac Newton, whose three laws of motion appeared in 1687. Friction was studied earlier by Leonardo da Vinci, and in the 1690s by Guillaume Amontons, who discovered that friction does not depend on the area of contact. The coefficient of friction μ is a model: real surfaces behave more subtly, but the model works well in many engineering calculations.

Otto's tip

Draw a large force diagram, mark every force with an arrow and a label, and resolve in two directions before writing equations.