Answers - Forces in two dimensions and friction

Year: Year 13

Answer to Question 1

Method: draw a clear force diagram, resolve every force into two perpendicular directions, and use $\cos$ for the component next to the angle and $\sin$ for the component opposite it.

a. The horizontal component is next to the 40° angle, so use cosine. Horizontal component $= 30\cos 40^\circ = 22.98 \approx 23.0$ N. So the horizontal component is about 23.0 N.
b. The vertical component is opposite the angle, so use sine. Vertical component $= 30\sin 40^\circ = 19.28 \approx 19.3$ N. So the vertical component is about 19.3 N.
c. Write the horizontal part with $\mathbf{i}$ and the vertical part with $\mathbf{j}$. $\mathbf{F} = (23.0\mathbf{i} + 19.3\mathbf{j})$ N. Check: $\sqrt{23.0^2 + 19.3^2} \approx 30$ N, the original magnitude.

Answer to Question 2

Method: draw a clear force diagram, resolve every force into two perpendicular directions, and use $\cos$ for the component next to the angle and $\sin$ for the component opposite it.

a. Resolve the 9 N force and add the components of both forces. Horizontal: $12 + 9\cos 60^\circ = 12 + 4.5 = 16.5$ N. Vertical: $9\sin 60^\circ = 7.79$ N. The 12 N force has no vertical component.
b. Use Pythagoras on the two total components. $R = \sqrt{16.5^2 + 7.794^2} = \sqrt{333} = 18.2$ N. So the resultant has magnitude about 18.2 N.
c. Use $\tan\theta = \dfrac{\text{vertical}}{\text{horizontal}}$. $\tan\theta = \dfrac{7.794}{16.5}$, so $\theta = 25.3^\circ$ above the x-axis. So the resultant acts at about 25.3° above the x-axis.

Answer to Question 3

a. Add the $\mathbf{i}$ components and the $\mathbf{j}$ components separately. $\mathbf{F} = (3\mathbf{i} + 8\mathbf{j})$ N. So the resultant is $(3\mathbf{i} + 8\mathbf{j})$ N.
b. Use Pythagoras for the magnitude and the inverse tangent for the direction. $|\mathbf{F}| = \sqrt{9 + 64} = \sqrt{73} = 8.54$ N. $\tan\theta = \dfrac{8}{3}$, so $\theta = 69.4^\circ$. So the resultant is about 8.54 N at 69.4° to $\mathbf{i}$.

c. Use Newton's second law $F = ma$ with the magnitude of the resultant. $a = \frac{F}{m} = \frac{8.544}{2} = 4.27$ m s^{-2}. So the particle accelerates at about 4.27 m s^{-2} in the direction of the resultant.

Answer to Question 4

a. The weight acts down; T_1 acts up and to the left at 40° to the horizontal; T_2 acts up and to the right at 50°.
b. Horizontal: $T_1 \cos 40^\circ = T_2 \cos 50^\circ$. Vertical: $T_1 \sin 40^\circ + T_2 \sin 50^\circ = 80$.
c. From the first equation, $T_1 = T_2 \frac{\cos 50^\circ}{\cos 40^\circ}$. Since $\cos 50^\circ = \sin 40^\circ$, $T_1 = T_2 \tan 40^\circ$. Substituting into the second, $T_2(\tan 40^\circ \sin 40^\circ + \sin 50^\circ) = 80$, i.e. $T_2(0.5393 + 0.7660) = 80$ so $T_2 = 61.3$ N and $T_1 = 61.28 \times 0.8391 = 51.4$ N.

Answer to Question 5

Method: friction can take any value up to the limiting value μR. The object moves only if the applied force is greater than μR; once it moves, friction equals μR and acts against the motion.

a. The floor is horizontal and there is no other vertical force, so the normal reaction balances the weight. $R = mg = 15 \times 9.8 = 147$ N. So $R = 147$ N.
b. The box starts to move when the push is larger than the limiting friction. Limiting friction $= \mu R = 0.4 \times 147 = 58.8$ N. A force just above 58.8 N will make the box move. So the least force is 58.8 N.
c. 100 N is more than 58.8 N, so the box moves and friction takes its limiting value. Apply $F = ma$. Resultant $= 100 - 58.8 = 41.2$ N. $a = \frac{41.2}{15} = 2.75$ m s^{-2}. So the box accelerates at about 2.75 m s^{-2}.

Answer to Question 6

Method: on a slope, resolve parallel and perpendicular to the plane. Perpendicular: $R = mg\cos\alpha$ (plus any other perpendicular components). Parallel: use $F = ma$, with friction $F = \mu R$ acting against the motion or the likely motion.

a. With no friction, the only force along the slope is the component of the weight. Along the slope: $mg \sin 25^\circ = ma$, so $a = 9.8 \sin 25^\circ = 4.14$ m s^{-2}. The mass cancels, so the acceleration does not depend on the mass.
b. First resolve perpendicular to find R, then use $F = ma$ along the slope with friction acting up the slope. Perpendicular to the slope: $R = mg\cos 25^\circ = 6 \times 9.8 \times 0.9063 = 53.3$ N. Along the slope: $mg \sin 25^\circ - 0.3R = 6a$, i.e. $24.85 - 15.99 = 6a$, so $a = 1.48$ m s^{-2}. So friction reduces the acceleration from 4.14 to about 1.48 m s^{-2}.

Answer to Question 7

a. Forces: weight 245 N down, normal reaction R up, friction F opposing motion, tension P along the rope at 30° above the horizontal.
b. Vertically (no acceleration): $R + P\sin 30^\circ = 25 \times 9.8 = 245$, so $R = 245 - 0.5P$.
c. Horizontally (constant speed): $P\cos 30^\circ = F = 0.15R = 0.15(245 - 0.5P)$. So $0.866P + 0.075P = 36.75$, giving $P = \dfrac{36.75}{0.9410} = 39.1$ N.
d. At 60° the vertical component is larger, so the normal reaction is smaller (less friction), but the horizontal component is smaller. The overall effect is an **increase** of P because $\cos 60^\circ$ is much smaller. (Check: $P = \dfrac{36.75}{0.5 + 0.1299} = 58.3$ N.)

Answer to Question 8

Method: draw a clear force diagram, resolve every force into two perpendicular directions, and use $\cos$ for the component next to the angle and $\sin$ for the component opposite it.

a. Each tension has a vertical component $T\sin 35^\circ$; the horizontal components cancel by symmetry. Resolving vertically: $2T\sin 35^\circ = 60$, so $T = \dfrac{30}{\sin 35^\circ}$. This is the required result.
b. Evaluate the formula. $T = \dfrac{30}{0.5736} = 52.3$ N. So each cable has a tension of about 52.3 N.
c. Write $T = \dfrac{30}{\sin\theta}$ and require $T \le 40$; a larger angle gives a smaller tension. $2T\sin\theta = 60$ with $T \le 40$ requires $\sin\theta \ge \dfrac{30}{40} = 0.75$, so $\theta \ge 48.6^\circ$. So the cables must make at least 48.6° with the horizontal.

Answer to Question 9

Method: on a slope, resolve parallel and perpendicular to the plane. Perpendicular: $R = mg\cos\alpha$ (plus any other perpendicular components). Parallel: use $F = ma$, with friction $F = \mu R$ acting against the motion or the likely motion.

a. In equilibrium the friction balances the component of the weight down the slope. $F = mg\sin 12^\circ = 900 \times 9.8 \times 0.2079 = 1834$ N. So the handbrake must provide about 1834 N of friction.
b. Resolve perpendicular to the slope. $R = mg\cos 12^\circ = 8820 \times 0.9781 = 8627$ N. So the normal reaction is about 8627 N.
c. Equilibrium needs $F \le \mu R$, so $\mu \ge \dfrac{F}{R}$. $\mu \ge \dfrac{F}{R} = \tan 12^\circ = 0.2126$. So the

coefficient of friction must be at least 0.213.

d. When the car slides, friction is limiting: $0.1R$ up the slope.
$mg\sin 12^\circ - 0.1R = 900a$: $1833.8 - 862.7 = 900a$, so $a = 1.08$ m s^{-2} down the slope. So the car slides down the slope with acceleration about 1.08 m s^{-2}.

Answer to Question 10

a. Weight 196 N down; push P along the handle (down and forward); normal reaction R up; friction opposing motion.

b. Vertically: $R = 196 + P\sin 35^\circ$.

c. Horizontally: $P\cos 35^\circ = 0.3R = 0.3(196 + P\sin 35^\circ)$. $P(0.8192 - 0.1721) = 58.8$, so $P = \dfrac{58.8}{0.6471} = 90.9$ N.

d. Pull the mower instead of pushing, or raise the handle angle, so the downward component of the force is smaller and the normal reaction (and friction) is reduced.

Answer to Question 11

Method: on a slope, resolve parallel and perpendicular to the plane. Perpendicular: $R = mg\cos\alpha$ (plus any other perpendicular components). Parallel: use $F = ma$, with friction $F = \mu R$ acting against the motion or the likely motion.

a. $R = mg\cos 20^\circ$. Down the slope: $mg\sin 20^\circ - 0.1mg\cos 20^\circ = ma$. So $a = 9.8(0.3420 - 0.0940) = 2.43$ m s^{-2}.

b. $v = at = 2.431 \times 5 = 12.2$ m s^{-1}.

c. $s = \frac{1}{2}at^2 = \frac{1}{2} \times 2.431 \times 25 = 30.4$ m.

d. Air resistance acts against the motion and increases with speed, so the acceleration would reduce and the skier would reach a smaller speed.

Answers: $a \approx 2.43$ m s^{-2}, $v \approx 12.2$ m s^{-1} after 5 s, and $s \approx 30.4$ m.

Answer to Question 12

a. Treat each crate separately with $F = ma$ in its own direction of motion; friction on the table crate is $0.25 \times 5g$. Hanging crate: $3g - T = 3a$. Table crate: $T - 0.25 \times 5g = 5a$. Both crates have the same acceleration a and the same tension T.

b. Add the two equations so that T cancels. Adding: $3g - 1.25g = 8a$, so $a = \dfrac{1.75 \times 9.8}{8} = 2.14$ m s^{-2}. So the system accelerates at about 2.14 m s^{-2}.

c. Substitute a into the equation of the hanging crate.
$T = 3(g - a) = 3(9.8 - 2.144) = 22.97 \approx 23.0$ N. Check with the table crate: $12.25 + 5 \times 2.144 = 22.97$ N.

d. The tension is the same throughout the string (light) and both crates have the

same acceleration (inextensible).

Answer to Question 13

Method: draw a clear force diagram, resolve every force into two perpendicular directions, and use $\cos$ for the component next to the angle and $\sin$ for the component opposite it.

a. Resolve each tension perpendicular to the centre line. Each rope has a sideways component $4000 \sin 20^\circ$ in opposite directions, so they cancel. So there is no sideways resultant and the barge moves along the centre line.
b. Resolve each tension along the centre line and add. Forward force $= 2 \times 4000 \cos 20^\circ = 7518$ N. So the total forward force is about 7518 N.
c. Subtract the resistance, then use $F = ma$. Net force $= 7518 - 6200 = 1318$ N, so $a = \dfrac{1318}{90000} = 0.0146$ m s^{-2}. So the barge accelerates at about 0.0146 m s^{-2}.

Answer to Question 14

Method: for connected vehicles, use the whole system to find the acceleration (the tension is internal and cancels), then use one vehicle alone to find the tension.

a. Total mass 2000 kg; total resistance $400 + 300 = 700$ N. Whole system: $3000 - 700 = 2000a$, so $a = 1.15$ m s^{-2}. So the acceleration is 1.15 m s^{-2}.
b. For the caravan alone, the tow bar pulls it forward and the resistance acts backwards. Caravan: $T - 300 = 800 \times 1.15$, so $T = 1220$ N. So the tension in the tow bar is 1220 N.
c. At a steady speed the resultant force on each part is zero. Steady speed means no acceleration: driving force $= 700$ N and for the caravan $T = 300$ N. So the engine only needs 700 N and the tow bar tension is 300 N.

Answer to Question 15

Method: on a slope, resolve parallel and perpendicular to the plane. Perpendicular: $R = mg \cos \alpha$ (plus any other perpendicular components). Parallel: use $F = ma$, with friction $F = \mu R$ acting against the motion or the likely motion.

a. The horizontal force has a component $P \sin 30^\circ$ into the plane, which adds to the weight component. Perpendicular to the plane: $R = mg \cos 30^\circ + P \sin 30^\circ$ with $mg = 98$. This is the required result.
b. If the block is about to move up, friction acts down the slope and is limiting. Up the plane: $P \cos 30^\circ = 98 \sin 30^\circ + 0.3R = 49 + 0.3(84.87 + 0.5P)$. So $0.8660P - 0.15P = 49 + 25.46$, $0.716P = 74.46$ and $P = 104$ N. So $P \approx 104$ N.
c. If the block is about to slide down, friction acts up the slope and is limiting. Down the plane: $P \cos 30^\circ + 0.3R = 98 \sin 30^\circ$: $0.866P + 25.46 + 0.15P = 49$, so $1.016P = 23.54$ and $P = 23.2$ N. So $P \approx 23.2$ N.

d. Between these two limiting cases friction is strong enough to hold the block. $23.2 \leq P \leq 104$ (3 s.f.).

Answer to Question 16

Method: draw a clear force diagram, resolve every force into two perpendicular directions, and use $\cos$ for the component next to the angle and $\sin$ for the component opposite it.

a. Resolve vertically and horizontally; the vertical component of the tension balances the weight. Vertically: $T\cos 25^\circ = 40$. Horizontally: $T\sin 25^\circ = H$. Dividing the vertical equation by $\cos 25^\circ$ gives the required result.

b. Evaluate T, then use the horizontal equation. $T = \dfrac{40}{0.9063} = 44.1$ N. $H = 44.13 \times 0.4226 = 18.7$ N, or $H = 40\tan 25^\circ = 18.7$ N. So $T \approx 44.1$ N and $H \approx 18.7$ N.

c. Dividing the two equations gives $H = 40\tan\theta$. $H = 40\tan 40^\circ = 33.6$ N. So a bigger angle needs a bigger horizontal force.

d. Use $T = \dfrac{40}{\cos\theta} \leq 100$. $T\cos\theta = 40$ and $T \leq 100$ require $\cos\theta \geq 0.4$, so $\theta \leq 66.4^\circ$. So the angle to the vertical must be at most 66.4°.

Answer to Question 17

a. The pull from the hanging mass is $3g = 29.4$ N. The component of the weight down the slope is $4g\sin 30^\circ = 19.6$ N and the maximum friction is $0.2 \times 4g\cos 30^\circ = 6.79$ N. Since $29.4 > 19.6 + 6.79$, the block moves up.

b. $R = 4g\cos 30^\circ = 33.95$. Hanging: $3g - T = 3a$. Block: $T - 4g\sin 30^\circ - 0.2R = 4a$. Adding: $29.4 - 19.6 - 6.79 = 7a$, so $a = 0.430$ m s^{-2}.

c. $T = 3(g - a) = 3(9.8 - 0.430) = 28.1$ N.

d. After 2 s the block has speed 0.86 m s^{-1} up the slope. With the string broken, the block decelerates with friction and gravity acting down the slope: $4a = -(19.6 + 6.79)$, $a = -6.60$ m s^{-2}. It stops after 0.13 s. Then $\tan 30^\circ = 0.577 > 0.2$ means it slides back down.

Answer to Question 18

Method: in equilibrium the vector sum of the forces is zero, so the $\mathbf{i}$ components and the $\mathbf{j}$ components each add to zero.

a. Equilibrium: $p + 2 - 4 = 0$ gives $p = 2$; $3 + q - 6 = 0$ gives $q = 3$.

b. Resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ is $(4\mathbf{i} + 6\mathbf{j})$ N, so $\mathbf{a} = \dfrac{4\mathbf{i} + 6\mathbf{j}}{0.5} = (8\mathbf{i} + 12\mathbf{j})$ m s^{-2}.

c. $|\mathbf{a}| = \sqrt{64 + 144} = \sqrt{208} = 14.4$ m s^{-2}, at $\tan^{-1}\dfrac{12}{8} = 56.3^\circ$ above $\mathbf{i}$.

d. $v = at = 14.42 \times 3 = 43.3$ m s^{-1}.

Answer to Question 19

a. At the point of sliding, limiting equilibrium gives $F = \mu R$ with $R = mg\cos\theta$ and $F = mg\sin\theta$. Dividing, $\mu = \tan\theta$.
b. Mean $= \dfrac{17 + 19 + 18 + 20 + 18}{5} = 18.4^\circ$, so $\mu \approx \tan 18.4^\circ = 0.333$.
c. $\tan 15^\circ = 0.268$, so friction can hold the block: it does **not** slide (and the mass does not matter).
d. Assumptions: the block is a particle and the coefficient of friction is constant over the surface. Improvement: repeat more trials, or use a smooth, slow rise of the board and a larger set of different blocks.

Answer to Question 20

a. $R = mg - P\sin\theta$ and $P\cos\theta = \mu R$. So $P\cos\theta = \mu mg - \mu P\sin\theta$, giving $P(\cos\theta + \mu\sin\theta) = \mu mg$.
b. P is smallest when the denominator $\cos\theta + \mu\sin\theta$ is largest. Differentiating: $-\sin\theta + \mu\cos\theta = 0$, so $\tan\theta = \mu$. (This is a maximum of the denominator, since the second derivative is $-\cos\theta - \mu\sin\theta$.)
c. $\theta = \tan^{-1} 0.4 = 21.8^\circ$. Denominator $= \cos 21.8^\circ + 0.4\sin 21.8^\circ = 0.9285 + 0.1486 = 1.0770$.
$P_{min} = \dfrac{0.4 \times 30 \times 9.8}{1.077} = 109$ N.

Answer: the crate is easiest to pull at about 21.8° above the horizontal, where the tension is about 109 N.

Answer to Question 21

Method: on a slope, resolve parallel and perpendicular to the plane. Perpendicular: $R = mg\cos\alpha$ (plus any other perpendicular components). Parallel: use $F = ma$, with friction $F = \mu R$ acting against the motion or the likely motion.

a. Equilibrium needs $F = mg\sin\alpha$ with $R = mg\cos\alpha$ and $F \le \mu R$. So $mg\sin\alpha \le \mu mg\cos\alpha$, i.e. $\tan\alpha \le \mu$. If $\tan\alpha > \mu$ the friction cannot be large enough and the block slides.
b. $\alpha_{max} = \tan^{-1} 0.45 = 24.2^\circ$.
c. $a = g(\sin 30^\circ - 0.45\cos 30^\circ) = 9.8(0.5 - 0.3897) = 1.08$ m s^{-2}.