Test answers - Differentiation: chain, product and quotient rules

Year: Year 13 · Duration: 40 minutes · /20

Answer to Question 1 (4 marks)

Mark scheme: 2 marks for each part.

a. Use $\frac{d}{dx}\ln u = \frac{u'}{u}$ with $u = 4x^2 + 1$, so $u' = 8x$. $\frac{dy}{dx} = \frac{8x}{4x^2+1}$ (1 for the chain rule, 1 for simplifying). So $\frac{dy}{dx} = \frac{8x}{4x^2+1}$.

b. Take $u = x^3$, $v = e^{-2x}$; the chain rule gives $v' = -2e^{-2x}$.
$\frac{dy}{dx} = 3x^2e^{-2x} - 2x^3e^{-2x} = x^2e^{-2x}(3-2x)$ (1 for each term, then factorising). So $a = 3$ and $b = -2$.

Common errors: in (a), writing $\frac{1}{4x^2+1}$ without multiplying by the derivative of the inside; in (b), forgetting the factor -2 from e^{-2x}, which gives the wrong bracket $(3-x)$. Check (b) at $x = 1$: $e^{-2}(3-2) = e^{-2} \approx 0.135$.

Answer to Question 2 (4 marks)

Mark scheme: 2 marks for the derivative, 1 mark for each stationary point.

a. Use the quotient rule with $u = x^2$ and $v = 2x + 1$.
$\frac{dy}{dx} = \frac{(2x+1)2x - x^2 \cdot 2}{(2x+1)^2} = \frac{2x^2+2x}{(2x+1)^2} = \frac{2x(x+1)}{(2x+1)^2}$ (1 for the method, 1 for the result). This is the required result.

b. A stationary point needs a zero numerator: $2x(x+1) = 0$, so $x = 0$ or $x = -1$.
$x = 0$: $(0, 0)$. $x = -1$: $y = \frac{1}{-1} = -1$: $(-1, -1)$. (1 mark each.) So the stationary points are $(0, 0)$ and $(-1, -1)$.

Answer to Question 3 (4 marks)

Mark scheme: 1 mark for checking the point, 1 mark for the implicit differentiation, 1 mark for the gradient, 1 mark for the tangent.

a. Substitute the point, then differentiate each term; $2xy$ needs the product rule.
$9 + 6 + 3 = 18$ so $(3, 1)$ lies on the curve (1). $2x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ (1), so $\frac{dy}{dx} = -\frac{x+y}{x+3y} = -\frac{4}{6} = -\frac{2}{3}$ (1). So the gradient at $(3, 1)$ is $-\frac{2}{3}$.

b. Use $y - y_1 = m(x - x_1)$ and multiply by 3. $y - 1 = -\frac{2}{3}(x - 3)$, i.e. $2x + 3y = 9$ (1). So the tangent is $2x + 3y = 9$.

Answer to Question 4 (4 marks)

Mark scheme: 2 marks for $\frac{dy}{dx}$; 1 mark for the point and gradient, 1 mark for the normal.

a. Find $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 6t^2 - 1$, then divide. $\frac{dy}{dx} = \frac{6t^2 - 1}{2t}$ (2). This is undefined at $t = 0$, where the tangent is vertical.

b. Substitute $t = 1$: $x = 3$, $y = 1$. The normal gradient is the negative reciprocal of $\frac{5}{2}$. At $t = 1$ the point is $(3, 1)$ and the gradient is $\frac{5}{2}$ (1). The normal is $y - 1 = -\frac{2}{5}(x - 3)$, i.e. $2x + 5y = 11$ (1). So the normal is $2x + 5y = 11$.

Answer to Question 5 (4 marks)

Method: link the rates by the chain rule. First find $\frac{dr}{dt}$ from the known rate of change of volume, then use it to find the rate of change of area.

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ (1). $100 = 100\pi \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{\pi} = 0.318$ cm s^{-1} (1). $\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$ (1) $= 8\pi \times 5 \times \frac{1}{\pi} = 40$ cm^2 s^{-1} (1).

Answer: when $r = 5$ cm, the radius increases at about 0.318 cm s^{-1} and the surface area at 40 cm² s^{-1}.