Lesson - Differentiation: chain, product and quotient rules

Year: Year 13

At A level you extend differentiation to exponentials, logarithms and trigonometric functions, and learn the chain, product and quotient rules, implicit and parametric differentiation.

Starter activity

A cyclist records her speed every minute, and a graph shows how it changes. How would you work out how fast her speed is changing at one moment? Now think of a more complicated situation: the speed is itself a product of two quantities that both change with time. Why might we need new rules to differentiate such functions?

Standard derivatives

Rule: standard derivatives (x in radians)

- $\frac{d}{dx}e^{kx} = ke^{kx}$, $\frac{d}{dx}a^x = a^x \ln a$, $\frac{d}{dx}\ln x = \frac{1}{x}$
- $\frac{d}{dx}\sin kx = k\cos kx$, $\frac{d}{dx}\cos kx = -k\sin kx$
- $\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\sec x = \sec x \tan x$

Note: why e^x

e^x is the only function equal to its own derivative, which is why it appears in growth and decay models.

The chain rule

Rule: chain rule

If $y = f(u)$ where $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. Differentiate the outside function, leaving the inside alone, then multiply by the derivative of the inside.

Worked example 1

Differentiate $(x^2+1)^5$, $\ln(x^2+1)$, e^{3x} and $\sin^2(3x)$.

$10x(x^2+1)^4$; $\dfrac{2x}{x^2+1}$; $3e^{3x}$. For $\sin^2(3x)$ nest: $y=u^2$, $u=\sin v$, $v=3x$, giving $6\sin 3x\cos 3x$.

The product rule

Rule: product rule

If $y=uv$ then $\dfrac{dy}{dx}=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$.

Worked example 2

Differentiate $y=x^2e^{3x}$ and find its stationary points.

With $u=x^2$, $v=e^{3x}$: $\dfrac{dy}{dx}=x^2\cdot 3e^{3x}+e^{3x}\cdot 2x=xe^{3x}(3x+2)$. This is zero when $x=0$ or $x=-\frac{2}{3}$ (since e^{3x} is never zero).

Worked example 3

Find the minimum of $y=x\ln x$.

$\dfrac{dy}{dx}=\ln x+1=0$ at $x=\frac{1}{e}$, where the minimum value is $-\frac{1}{e}$.

The quotient rule

Rule: quotient rule

If $y=\dfrac{u}{v}$ then $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$.

Worked example 4

Differentiate $y=\dfrac{e^x}{x+1}$ and prove $\dfrac{d}{dx}\tan x=\sec^2 x$.

$\dfrac{dy}{dx}=\dfrac{(x+1)e^x-e^x}{(x+1)^2}=\dfrac{xe^x}{(x+1)^2}$, so for $x>-1$ there is a minimum at $x=0$. For tangent: $\dfrac{\cos x\cos x+\sin x\sin x}{\cos^2 x}=\sec^2 x$.

Watch out

The numerator of the quotient rule is $vu' - uv'$, in that order. In the chain rule never forget the inside derivative: $\frac{d}{dx}e^{3x} = e^{3x}$ is wrong.

Implicit differentiation

Method: implicit differentiation

1. Differentiate every term with respect to x.
2. A term in y uses the chain rule: $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$. A product xy uses the product rule: $y + x\frac{dy}{dx}$.
3. Collect the $\frac{dy}{dx}$ terms and solve.

Worked example 5

Find $\frac{dy}{dx}$ for $x^2 + 3xy + y^2 = 5$.

$2x + 3y + 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{2x+3y}{3x+2y}$. (For $x^2 + y^2 = 25$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{3}{4}$ at $(3, 4)$.)

Worked example 6

Differentiate $y = x^x$.

Take logarithms: $\ln y = x \ln x$, so $\frac{1}{y}\frac{dy}{dx} = \ln x + 1$ and $\frac{dy}{dx} = x^x(\ln x + 1)$.

Parametric differentiation

Rule: parametric differentiation

For $x = f(t)$, $y = g(t)$: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Worked example 7

For $x = t^2$, $y = t^3 - 3t$ find $\frac{dy}{dx}$ at $t = 2$ and the stationary points.

$\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t}$, which is $\frac{9}{4}$ at $t = 2$ and zero at $t = \pm 1$.

Tangents, normals and rates of change

Worked example 8

Find the normal to $y = e^{2x}$ at $(0, 1)$, and the tangent to $x = 2t$, $y = t^2$ at $t = 3$.

The tangent gradient of e^{2x} is 2, so the normal is $x + 2y = 2$. For the parametric curve $\dfrac{dy}{dx} = \dfrac{2t}{2} = 3$ at $(6, 9)$: $y = 3x - 9$.

Worked example 9

The radius of a sphere grows at $2\ \text{cm}\,\text{s}^{-1}$. Find $\dfrac{dV}{dt}$ when $r = 5$.

$\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2 \times 2 = 200\pi \approx 628\ \text{cm}^3\,\text{s}^{-1}$.

Differential equations: forming and checking

Worked example 10

Show that $y = Ae^{3x} + Be^{-x}$ satisfies $y'' - 2y' - 3y = 0$.

$y' = 3Ae^{3x} - Be^{-x}$ and $y'' = 9Ae^{3x} + Be^{-x}$. For the A terms $9 - 6 - 3 = 0$, for the B terms $1 + 2 - 3 = 0$. "Population grows at a rate proportional to its size" becomes $\dfrac{dP}{dt} = kP$.

Choosing the right rule

Method: choosing the rule

1. Look at the last operation. A sum: differentiate term by term.
2. A product of two functions of x: product rule. A division: quotient rule (or a negative power if the numerator is constant).
3. A function applied to an inner expression ($\sin$, $\ln$, a power): chain rule.
4. Combined cases need two rules, for example $x^2 e^{3x}$ or $\dfrac{\sin 2x}{x^2}$. Write u, v, u', v' separately.

Note: using the derivative

Factorise $\frac{dy}{dx}$ before setting it to zero. e^{kx} is never zero and $\ln x$ needs $x > 0$, so reject values outside the domain. Classify with $\frac{d^2y}{dx^2}$ or the sign of the gradient either side. In context, give units and interpret the sign of the derivative.

Watch out

Write $\frac{dy}{dx}$ when differentiating a y-term implicitly. Use radians. Factorise the derivative before setting it to zero.

Worked example 11: A chain rule inside a product

Differentiate $y = e^{2x}\sin 3x$ and find the smallest positive x for which $\frac{dy}{dx} = 0$.

Let $u = e^{2x}$, $v = \sin 3x$: $u' = 2e^{2x}$, $v' = 3\cos 3x$. $\frac{dy}{dx} = e^{2x}(2\sin 3x + 3\cos 3x)$. Setting this to zero: $\tan 3x = -\frac{3}{2}$, so $3x = \pi - 0.9828 = 2.1588$ and $x = 0.720$. Remember e^{2x} is never zero.

Worked example 12: Differentiating a power of a function with the quotient rule

Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

Write $\sec x = \frac{1}{\cos x}$. Quotient rule with $u = 1$, $v = \cos x$: $\frac{dy}{dx} = \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x$. Alternatively use the chain rule on $(\cos x)^{-1}$.

Worked example 13: Differentiating an exponential with base a

Show that $\frac{d}{dx}(2^x) = 2^x \ln 2$ and find the gradient of $y = 2^x$ at $x = 3$.

Write $2^x = e^{x\ln 2}$. By the chain rule the derivative is $\ln 2 \times e^{x\ln 2} = 2^x \ln 2$. At $x = 3$: $8\ln 2 = 5.55$.

Worked example 14: A second parametric derivative

For $x = t^2$, $y = t^3 - 3t$ show that the curve has a minimum at $t = 1$.

$\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t} = \dfrac{3}{2}\left(t - \dfrac{1}{t}\right)$. This is zero at $t = 1$. Differentiating again with respect to x: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \div \dfrac{dx}{dt} = \dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right) \div 2t$. At $t = 1$ this is $\dfrac{3}{2} \times 2 \div 2 = 1.5 > 0$, so it is a minimum.

Worked example 15: Logarithmic differentiation of a product

Differentiate $y = \dfrac{x^2\sqrt{x+1}}{e^{3x}}$ using logarithms.

$\ln y = 2\ln x + \frac{1}{2}\ln(x+1) - 3x$. Then $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2}{x} + \dfrac{1}{2(x+1)} - 3$, so $\dfrac{dy}{dx} = y\left(\dfrac{2}{x} + \dfrac{1}{2(x+1)} - 3\right)$. This avoids a long mixture of the product and quotient rules.

Worked example 16: Which rule?

State the rule needed for (a) $\ln(\cos x)$, (b) $x\ln x$, (c) $\dfrac{x}{e^{2x}}$ and differentiate (a).

(a) chain rule: $\dfrac{-\sin x}{\cos x} = -\tan x$. (b) product rule. (c) quotient rule, or a product with e^{-2x}. The last operation tells you the rule: a function applied to a function is the chain rule; two functions multiplied is the product rule; one divided by another is the quotient rule.

Rule: derivatives of reciprocal trigonometric functions

Using the quotient rule and the chain rule, $\dfrac{d}{dx}(\sec x) = \sec x \tan x$, $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ and $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$. These formulae are in the formula booklet in some specifications, but you must be able to derive them. With a multiple of x, remember the chain rule: $\dfrac{d}{dx}\tan 3x = 3\sec^2 3x$.

Worked example 17: Gradient of a tangent from a product

Find the gradient of $y = x\sqrt{2x+1}$ at $x = 4$.

Write $y = x(2x+1)^{1/2}$. Product and chain rule: $\dfrac{dy}{dx} = (2x+1)^{1/2} + x \cdot \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = \sqrt{2x+1} + \dfrac{x}{\sqrt{2x+1}}$. At $x = 4$: $3 + \dfrac{4}{3} = \dfrac{13}{3} = 4.33$. Notice that two different rules are used inside one derivative;

label each step so that the marker can follow it.

Worked example 18: A normal that meets the axes

The curve $y = e^{2x}$ meets the y-axis at $A(0, 1)$. Find where the normal at A meets the x-axis.

$\frac{dy}{dx} = 2e^{2x} = 2$ at $x = 0$, so the normal has gradient $-\frac{1}{2}$: $y = 1 - \frac{1}{2}x$. It meets the x-axis when $y = 0$, so $x = 2$, at the point $(2, 0)$. The area of the triangle formed by the normal and the two axes is $\frac{1}{2} \times 2 \times 1 = 1$.

Worked example 19: A related rates problem with a ladder-style relation

A point moves along the curve $y = x^2$ so that $\frac{dx}{dt} = 3$ units per second. Find $\frac{dy}{dt}$ when $x = 2$.

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = 2x \times 3 = 6x$. At $x = 2$ this is 12 units per second. The chain rule links the rates: we know how fast x changes and the gradient of the curve, and we need how fast y changes.

Worked example 20: A product of three factors

Differentiate $y = x^2 e^x \sin x$.

Treat $y = (x^2 e^x) \sin x$. First $\frac{d}{dx}(x^2 e^x) = e^x(x^2 + 2x)$. Then $\frac{dy}{dx} = e^x(x^2 + 2x)\sin x + x^2 e^x \cos x = xe^x\left[(x + 2)\sin x + x\cos x\right]$. Taking out the common factor xe^x makes it much easier to solve $\frac{dy}{dx} = 0$.

Worked example 21: Forming and solving a rate equation

The mass M grams of a radioactive sample satisfies $\frac{dM}{dt} = -0.04M$. Show that $M = M_0 e^{-0.04t}$ is a solution and find the time for the mass to halve.

Differentiate: $\frac{dM}{dt} = -0.04M_0 e^{-0.04t} = -0.04M$, so the equation is satisfied for any M_0. Halving: $e^{-0.04t} = \frac{1}{2}$, so $t = \frac{\ln 2}{0.04} = 17.3$ units of time. The half-life does not depend on M_0.

Worked example 22: Using implicit differentiation with an exponential

Find the gradient of $e^y + xy = e$ at the point $(0, 1)$.

Differentiate each term: $e^y \dfrac{dy}{dx} + y + x\dfrac{dy}{dx} = 0$. At $(0, 1)$: $e\dfrac{dy}{dx} + 1 = 0$, so $\dfrac{dy}{dx} = -\dfrac{1}{e} = -0.368$. The derivative of e^y with respect to x needs the chain rule, giving the extra factor $\dfrac{dy}{dx}$.

Worked example 23: Why a model has to be questioned

A model says the height of a plant is $h = \dfrac{120t}{t + 10}$ cm after t days. Find the growth rate and comment on the model for large t.

Quotient rule: $\dfrac{dh}{dt} = \dfrac{(t + 10) \cdot 120 - 120t \cdot 1}{(t + 10)^2} = \dfrac{1200}{(t + 10)^2}$. This is always positive, so the plant always grows, but the rate tends to 0 and $h \to 120$ cm. This agrees with a plant that has a maximum height. A model with $h \to \infty$ would be unrealistic. At $t = 10$ the rate is 3 cm per day.

Note: the shape of a product

The graph below shows $y = x^2 e^{-x}$. Differentiating with the product rule gives $y\prime = xe^{-x}(2 - x)$. The stationary points are at $x = 0$ (minimum) and $x = 2$ (maximum), and the curve decays towards the axis.



Method: a stationary point problem

1. Identify the rule you need and write u, v, $u\prime$, $v\prime$ in a small table.
2. Differentiate and simplify; factorise as far as possible.
3. Set $\dfrac{dy}{dx} = 0$; remember that e^{kx} is never zero and $\ln x$ needs $x > 0$.
4. Find the y-values, then classify using the second derivative or the sign of the gradient either side.
5. Answer in context with units.

Watch out: frequent errors

- Forgetting the derivative of the inside: $\frac{d}{dx}\sin(3x) = 3\cos 3x$, not $\cos 3x$.
- In the quotient rule, writing the numerator as $uv\prime - vu\prime$ instead of $vu\prime - uv\prime$.
- Writing $\frac{d}{dx}(y^2) = 2y$ instead of $2y\frac{dy}{dx}$.
- Using degrees: all the derivative formulae for sine and cosine need radians.
- For parametric curves, forgetting that $\frac{dy}{dx}$ is a ratio of derivatives with respect to t, so the answer is usually in terms of t.

Did you know?

The product rule is usually credited to Gottfried Leibniz, who in 1675 first wrote it down. At first he made a mistake and thought the derivative of a product was simply the product of the derivatives! The number e, whose exponential function is its own derivative, is named after Leonhard Euler, and it appears whenever a quantity grows or decays at a rate proportional to its size.

Otto's tip

Label u, v, u' and v' in a small table before you use the product or quotient rule.