Worksheet - Differentiation: chain, product and quotient rules

Year: Year 13 · 21

Show your working.

Question 1

Question 1 – The chain rule *(Section A · Skills practice)*

Differentiate each function with respect to x.

a. $y = (2x + 1)^5$ [2 marks]
b. $y = e^{-4x}$ [1 mark]
c. $y = \ln(5x^2 + 1)$ [2 marks]
d. $y = \sin(3x^2)$ [2 marks]

Question 2

Question 2 – The product rule *(Section A · Skills practice)*

Use the product rule to differentiate each function.

a. $y = x^3 e^{2x}$ [3 marks]
b. $y = x^2 \ln x$ [3 marks]
c. $y = x \sin 3x$ [3 marks]

Question 3

Question 3 – The quotient rule *(Section A · Skills practice)*

Use the quotient rule to differentiate each function and simplify.

a. $y = \dfrac{2x + 1}{x^2 + 3}$ [3 marks]
b. $y = \dfrac{e^x}{x}$ [3 marks]
c. $y = \dfrac{\sin x}{x^2}$ [3 marks]

Question 4

Question 4 – Implicit differentiation *(Section A · Skills practice)*

A curve has equation $x^2 + xy + y^2 = 7$.

a. Show that $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$. [4 marks]
b. Verify that $(1, 2)$ lies on the curve and find the gradient there. [2 marks]
c. Find the equation of the normal at $(1, 2)$. [3 marks]

Question 5

Question 5 – Parametric differentiation *(Section A · Skills practice)*

A curve is given by $x = 2t + 1$, $y = t^3 - 12t$.

a. Find $\frac{dy}{dx}$ in terms of t. [2 marks]
b. Find the coordinates of the stationary points. [4 marks]
c. Find the gradient at $t = 3$. [1 mark]

Question 6

Question 6 – Tangent and normal *(Section A · Skills practice)*

The curve C has equation $y = \ln(3x - 2)$.

a. Find $\frac{dy}{dx}$. [2 marks]
b. Find the equation of the tangent to C at the point where $x = 1$. [3 marks]
c. Find the equation of the normal at the same point, in the form $ax + by = c$. [3 marks]

Question 7

Question 7 – Growing bacteria *(Section B · Real-life practice)*

A scientist grows a colony of bacteria in a laboratory. The number of bacteria P at time t hours is modelled by $P = 200e^{0.3t}$.

a. Find the initial number of bacteria. [1 mark]
b. Find $\frac{dP}{dt}$ and show that $\frac{dP}{dt} = 0.3P$. [2 marks]
c. Find the rate of growth when $t = 4$. [2 marks]
d. Find the time at which the colony is growing at 100 bacteria per hour. [3 marks]

Question 8

Question 8 – A cooling drink *(Section B · Real-life practice)*

A cup of tea is left to cool in a cafe. Its temperature θ degrees Celsius after t minutes is modelled by $\theta = 20 + 70e^{-0.05t}$.

a. Find the initial temperature and the temperature the tea approaches in the long term. [2 marks]
b. Find $\frac{d\theta}{dt}$ and interpret its sign. [2 marks]
c. Show that $\frac{d\theta}{dt} = -0.05(\theta - 20)$. [2 marks]
d. Find the rate of cooling when $t = 10$. [2 marks]

Question 9

Question 9 – Drug concentration *(Section B · Real-life practice)*

After a patient takes a tablet, the concentration of a drug in the blood is modelled by $C = 12te^{-0.5t}$ milligrams per litre, where t is the time in hours after taking the tablet, $t \geq 0$.

t	0	1	2	4	6	8
C	0	7.28	8.83	6.50	3.58	1.76

a. Show that $\frac{dC}{dt} = 6e^{-0.5t}(2 - t)$. [3 marks]
b. Find the time at which the concentration is greatest, and the maximum concentration. [4 marks]
c. The drug is only effective while $C > 5$. Use the table to estimate for how long this is. [2 marks]

Question 10

Question 10 – Ice cream sales *(Section B · Real-life practice)*

An ice cream seller models the number of ice creams sold per hour, S, at time t hours after opening on a sunny day by $S = \frac{400t}{t^2 + 4}$, $t \geq 0$.

a. Show that $\frac{dS}{dt} = \frac{400(4 - t^2)}{(t^2 + 4)^2}$. [4 marks]
b. Find the time at which sales per hour are greatest and the maximum value. [3 marks]
c. Find the sales rate when $t = 6$ and explain why the model might not be realistic for large t. [3 marks]

Question 11

Question 11 – Inflating a balloon *(Section B · Real-life practice)*

A spherical party balloon is inflated so that its radius increases at a constant rate of 0.5 cm per second. The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and its surface area is $A = 4\pi r^2$.

a. Find $\frac{dV}{dt}$ when $r = 6$. [3 marks]
b. Find $\frac{dA}{dt}$ when $r = 6$. [3 marks]
c. Explain why $\frac{dV}{dt}$ is not constant even though $\frac{dr}{dt}$ is constant. [1 mark]

Question 12

Question 12 – A leaking cone *(Section B · Real-life practice)*

Water leaks from a conical funnel held with its vertex downwards. The depth of water is h cm and the radius of the water surface is $\frac{h}{2}$ cm. The volume of water is $V = \frac{1}{3}\pi r^2 h$. Water leaks out at 20 cm^3 s^{-1}.

a. Show that $V = \frac{\pi h^3}{12}$. [2 marks]
b. Find $\frac{dh}{dt}$ when $h = 10$. [4 marks]
c. Explain why the depth falls more quickly as the funnel empties. [1 mark]

Question 13

Question 13 – A roller coaster track *(Section B · Real-life practice)*

The path of a roller coaster is modelled by $x = 4t$, $y = t^3 - 6t^2 + 9t$ for $0 \le t \le 4$, where x and y are measured in metres and t is a parameter.

a. Find $\frac{dy}{dx}$ in terms of t. [3 marks]
b. Find the coordinates of the highest and lowest points of the track for $0 \le t \le 4$ at which the track is horizontal. [4 marks]
c. Find the gradient of the track at the end $t = 4$. [2 marks]

Question 14

Question 14 – A swinging pendulum *(Section B · Real-life practice)*

The horizontal displacement of a pendulum bob from its rest position is $x = 5\cos 2t$ centimetres, where t is the time in seconds.

a. Find the velocity $v = \frac{dx}{dt}$. [2 marks]
b. Find the acceleration $a = \frac{d^2x}{dt^2}$ and show that $a = -4x$. [3 marks]
c. Find the first time at which the bob is at rest momentarily. [2 marks]
d. Find the maximum speed. [1 mark]

Question 15

Question 15 – Stationary points of x squared times e to minus x *(Section C · Problem solving)*

A curve has equation $y = x^2 e^{-x}$ for $x \geq 0$. It models the height (in metres) of a firework x seconds after launch scaled for the display.

x	0	1	2	3	4	6
y	0	0.37	0.54	0.45	0.29	0.09

a. Show that $\frac{dy}{dx} = xe^{-x}(2 - x)$. [3 marks]

b. Find the coordinates of the stationary points and determine their nature. [5 marks]

c. State the range of y and explain what happens as $x \to \infty$. [2 marks]

d. Find the second derivative and use it to confirm the maximum. [3 marks]

Question 16

Question 16 – Horizontal tangents on an ellipse-like curve *(Section C · Problem solving)*

A curve has equation $x^2 + xy + y^2 = 7$.

a. Show that the tangent is horizontal where $y = -2x$. [3 marks]

b. Find the coordinates of the points where the tangent is horizontal. [4 marks]

c. Find the coordinates of the points where the tangent is vertical. [4 marks]

Question 17

Question 17 – An ellipse in parametric form *(Section C · Problem solving)*

A curve is given by $x = 3\cos t$, $y = 2\sin t$ for $0 \leq t$.

a. Show that $\frac{dy}{dx} = -\frac{2}{3}\cot t$. [3 marks]

b. Find the equation of the tangent at $t = \frac{\pi}{4}$. [4 marks]

c. Show that the Cartesian equation of the curve is $\frac{x^2}{9} + \frac{y^2}{4} = 1$. [2 marks]

d. Find the points where the tangent is parallel to the y-axis. [2 marks]

Question 18

Question 18 – Checking a differential equation solution *(Section C · Problem solving)*

The displacement y of a spring-like system satisfies the differential equation

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$

a. Show that $y = Ae^{2x} + Be^{-3x}$ is a solution for any constants A and B. [4 marks]
b. Given that $y = 1$ and $\frac{dy}{dx} = -8$ when $x = 0$, find A and B. [4 marks]
c. Find the value of y when $x = 1$. [1 mark]
d. Describe what happens to y as $x \to \infty$. [1 mark]

Question 19

Question 19 – Minimum surface area of a can *(Section C · Problem solving)*

A manufacturer makes an open-topped cylindrical recycling container for glass of volume 500 cm^3 with radius r cm and height h cm. The total surface area (base and curved side) is $A = \pi r^2 + 2\pi rh$.

a. Show that $A = \pi r^2 + \frac{1000}{r}$. [3 marks]
b. Find $\frac{dA}{dr}$ and the value of r for which A is stationary. [4 marks]
c. Show that this gives a minimum and find the minimum area and the corresponding height. [4 marks]

Question 20

Question 20 – Challenge: a variable exponent *(Section D · Challenge)*

Let $y = x^{\sin x}$ for $x > 0$.

a. By taking natural logarithms, show that $\frac{dy}{dx} = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$. [4 marks]
b. Find the exact value of $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$. [3 marks]

Question 21

Question 21 – Challenge: tangents through the origin *(Section D · Challenge)*

Find the equation of the tangent from the origin to each curve.

a. $y = e^x$ [5 marks]
b. $y = \ln x$ [5 marks]