Answers - Differentiation: chain, product and quotient rules

Year: Year 13

Answer to Question 1

Method (chain rule): if $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)) \times g'(x)$. Differentiate the outside function, keep the inside unchanged, then multiply by the derivative of the inside.

a. Outside: power 5; inside: $2x + 1$. $\frac{dy}{dx} = 5(2x+1)^4 \times 2 = 10(2x+1)^4$. The final answer is $10(2x+1)^4$.

b. The inside function is $-4x$, whose derivative is -4; the exponential is unchanged. $\frac{dy}{dx} = -4e^{-4x}$. So the gradient is always negative.

c. The derivative of $\ln u$ is $\frac{u'}{u}$ with $u = 5x^2 + 1$. $\frac{dy}{dx} = \frac{1}{5x^2+1} \times 10x = \frac{10x}{5x^2+1}$. So $\frac{dy}{dx} = \frac{10x}{5x^2+1}$.

d. The outside function is $\sin$, which becomes $\cos$; the inside $3x^2$ gives the factor $6x$. $\frac{dy}{dx} = \cos(3x^2) \times 6x = 6x\cos(3x^2)$. So $\frac{dy}{dx} = 6x\cos(3x^2)$.

Answer to Question 2

Method (product rule): if $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. Write down u, v, u' and v' first, then substitute and factorise.

a. The derivative of e^{2x} uses the chain rule. $u = x^3$, $v = e^{2x}$, $u\prime = 3x^2$, $v\prime = 2e^{2x}$. $\frac{dy}{dx} = x^3 \cdot 2e^{2x} + e^{2x} \cdot 3x^2 = x^2e^{2x}(2x+3)$. Factorising x^2e^{2x} gives the neatest form.

b. The derivative of $\ln x$ is $\frac{1}{x}$. $u = x^2$, $v = \ln x$: $\frac{dy}{dx} = x^2 \cdot \frac{1}{x} + 2x\ln x = x + 2x\ln x = x(1 + 2\ln x)$. So $\frac{dy}{dx} = x(1+2\ln x)$.

c. The derivative of $\sin 3x$ is $3\cos 3x$ by the chain rule. $u = x$, $v = \sin 3x$: $\frac{dy}{dx} = x \cdot 3\cos 3x + \sin 3x = 3x\cos 3x + \sin 3x$. So $\frac{dy}{dx} = 3x\cos 3x + \sin 3x$.

Answer to Question 3

Method (quotient rule): if $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\,u' - u\,v'}{v^2}$. The order in the numerator matters: start with the denominator times the derivative of the numerator.

a. Here $u = 2x + 1$, $u' = 2$, $v = x^2 + 3$, $v' = 2x$. Expand the numerator carefully. $\frac{dy}{dx} = \frac{(x^2+3)\cdot 2 - (2x+1)\cdot 2x}{(x^2+3)^2} = \frac{2x^2 + 6 - 4x^2 - 2x}{(x^2+3)^2} = \frac{-2x^2 - 2x + 6}{(x^2+3)^2}$. The numerator can also be written $-2(x^2 + x - 3)$.

b. Here $u = e^x$, $u' = e^x$, $v = x$, $v' = 1$. $\frac{dy}{dx} = \frac{xe^x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$. So the gradient is zero only when $x = 1$.

c. Here $u = \sin x$, $v = x^2$. After using the rule, divide the top and bottom by x. $\frac{dy}{dx} = \frac{x^2\cos x - \sin x \cdot 2x}{x^4} = \frac{x\cos x - 2\sin x}{x^3}$. So $\frac{dy}{dx} = \frac{x\cos x - 2\sin x}{x^3}$.

Answer to Question 4

Method: in implicit differentiation, differentiate each term with respect to x; a term in y gains a factor $\frac{dy}{dx}$, and xy needs the product rule.

a. Differentiate with respect to x: $2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$. Collect: $(x + 2y)\frac{dy}{dx} = -(2x + y)$, so $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$. This is the required result.

b. Substitute $x = 1$, $y = 2$ into the equation, then into the formula for the gradient. $1 + 2 + 4 = 7$ so the point lies on the curve. Gradient $= -\frac{2+2}{1+4} = -\frac{4}{5}$. So the gradient of the tangent at $(1, 2)$ is $-\frac{4}{5}$.

c. The normal is perpendicular to the tangent, so its gradient is the negative reciprocal. The normal gradient is $\frac{5}{4}$. $y - 2 = \frac{5}{4}(x - 1)$, so $4y - 8 = 5x - 5$, i.e. $5x - 4y + 3 = 0$. So the normal is $5x - 4y + 3 = 0$.

Answer to Question 5

Method: for a parametric curve, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. The tangent is horizontal when $\frac{dy}{dt} = 0$ and vertical when $\frac{dx}{dt} = 0$.

a. Differentiate x and y separately with respect to t, then divide. $\frac{dx}{dt} = 2$ and

$\frac{dy}{dt} = 3t^2 - 12$, so $\frac{dy}{dx} = \frac{3t^2 - 12}{2}$. This simplifies to $\frac{3}{2}(t^2 - 4)$.

b. Set the gradient to zero, find t, then substitute each value into x and y. $\frac{dy}{dx} = 0$ when $t^2 = 4$, $t = \pm 2$. At $t = 2$: $(5, -16)$. At $t = -2$: $(-3, 16)$. So the stationary points are $(5, -16)$ and $(-3, 16)$.

c. Substitute $t = 3$ into the formula from part a). At $t = 3$: $\frac{27 - 12}{2} = 7.5$. So the gradient at $t = 3$ is 7.5.

Answer to Question 6

Method (chain rule): if $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)) \times g'(x)$. Differentiate the outside function, keep the inside unchanged, then multiply by the derivative of the inside.

a. Use $\frac{d}{dx}\ln(u) = \frac{u'}{u}$ with $u = 3x - 2$. $\frac{dy}{dx} = \frac{3}{3x - 2}$. This is defined only for $x > \frac{2}{3}$.

b. Find the point and the gradient at $x = 1$, then use $y - y_1 = m(x - x_1)$. At $x = 1$, $y = \ln 1 = 0$ and the gradient is $\frac{3}{1} = 3$. Tangent: $y = 3(x - 1) = 3x - 3$. So the tangent is $y = 3x - 3$.

c. The normal gradient is the negative reciprocal of 3. Normal gradient $= -\frac{1}{3}$. $y = -\frac{1}{3}(x - 1)$, so $3y = -x + 1$, i.e. $x + 3y = 1$. So the normal is $x + 3y = 1$.

Answer to Question 7

a. Substitute $t = 0$, remembering that $e^0 = 1$. At $t = 0$, $P = 200$. So the colony starts with 200 bacteria.

b. Differentiate the exponential with the chain rule. $\frac{dP}{dt} = 200 \times 0.3e^{0.3t} = 60e^{0.3t} = 0.3P$. So the rate of growth is proportional to the size of the colony.

c. Substitute $t = 4$ into $\frac{dP}{dt}$. $\frac{dP}{dt} = 60e^{1.2} = 199$ bacteria per hour. So after 4 hours the colony grows by about 199 bacteria per hour.

d. Set $\frac{dP}{dt} = 100$ and solve with natural logarithms. $60e^{0.3t} = 100$ gives $e^{0.3t} = \frac{5}{3}$, so $t = \frac{\ln(5/3)}{0.3} = 1.70$ hours. So this happens after about 1.70 hours, which is 1 hour 42 minutes.

Answer to Question 8

a. Initially $\theta = 20 + 70 = 90°$C. As $t \to \infty$, $e^{-0.05t} \to 0$ so $\theta \to 20°$C (room temperature). So the tea starts at 90 °C and cools towards 20 °C.

b. Use the chain rule: $70 \times (-0.05) = -3.5$. $\dfrac{d\theta}{dt} = -3.5e^{-0.05t}$, which is always negative, so the tea is always cooling.

c. Express $e^{-0.05t}$ in terms of θ. $\theta - 20 = 70e^{-0.05t}$, so $-0.05(\theta - 20) = -3.5e^{-0.05t} = \dfrac{d\theta}{dt}$. This is Newton's law of cooling: the rate of cooling is proportional to the difference from room temperature.

d. Substitute $t = 10$. $-3.5e^{-0.5} = -2.12$, so the tea is cooling at $2.12°$C per minute.

Answer to Question 9

a. Product rule: $\dfrac{dC}{dt} = 12e^{-0.5t} + 12t(-0.5)e^{-0.5t} = 12e^{-0.5t}(1 - 0.5t) = 6e^{-0.5t}(2 - t)$.

b. $\dfrac{dC}{dt} = 0$ when $t = 2$ (as $e^{-0.5t} > 0$). The gradient is positive for t and negative for $t > 2$, so this is a maximum. $C(2) = 24e^{-1} = 8.83$ mg/l.

c. $C > 5$ from about $t = 0.5$ to about $t = 5$ (reading between table values: $C(1) = 7.28$, $C(4) = 6.50$, $C(6) = 3.58$), so roughly 4 to 5 hours. Solving exactly gives 0.55, about 4.4 hours.

Answer: the concentration is greatest after 2 hours, at about 8.83 mg/l, and the drug is effective for about 4.4 hours.

Answer to Question 10

a. Quotient rule:
$\dfrac{dS}{dt} = \dfrac{(t^2+4)\cdot 400 - 400t \cdot 2t}{(t^2+4)^2} = \dfrac{400t^2 + 1600 - 800t^2}{(t^2+4)^2} = \dfrac{400(4 - t^2)}{(t^2+4)^2}$.

b. $\dfrac{dS}{dt} = 0$ when $t = 2$ ($t \geq 0$). The numerator changes from positive to negative, so a maximum. $S(2) = \dfrac{800}{8} = 100$ ice creams per hour.

c. $S(6) = \dfrac{2400}{40} = 60$. As $t \to \infty$, $S \to 0$: sales fall to zero in the long run, but in reality the seller closes, the weather changes and demand depends on the time of day.

Answer to Question 11

Method: connected rates of change are linked by the chain rule, for example $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$. Find the derivative of the formula first, then substitute the given values at the end.

a. Here $\frac{dV}{dr} = 4\pi r^2$. $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 0.5 = 4\pi \times 36 \times 0.5 = 72\pi = 226$ cm ^{3} s^{-1}. So the volume grows at about 226 cm³ per second at that instant.

b. Here $\frac{dA}{dr} = 8\pi r$. $\frac{dA}{dt} = 8\pi r \times 0.5 = 8\pi \times 6 \times 0.5 = 24\pi = 75.4$ cm^2 s^{-1}. So the surface area grows at about 75.4 cm² per second.

c. Look at the formula for $\frac{dV}{dt}$. $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ depends on r, which keeps growing, so the volume increases ever faster. So a larger balloon needs more air for the same increase in radius.

Answer to Question 12

Method: connected rates of change are linked by the chain rule, for example $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$. Find the derivative of the formula first, then substitute the given values at the end.

a. Replace r by $\frac{h}{2}$ so that V depends on h only. $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$. This is the required formula.

b. The volume decreases, so $\frac{dV}{dt} = -20$. $\frac{dV}{dh} = \frac{\pi h^2}{4}$. $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$ gives $-20 = \frac{\pi \times 100}{4}\frac{dh}{dt}$, so $\frac{dh}{dt} = -\frac{80}{100\pi} = -0.255$ cm s^{-1}. So the depth falls by about 0.255 cm per second when the water is 10 cm deep.

c. Look at how $\frac{dh}{dt}$ depends on h. $\frac{dh}{dt} = -\frac{80}{\pi h^2}$: as h falls the funnel narrows, so each cm^3 lost corresponds to a larger drop in depth. When h is halved, the depth falls four times as fast.

Answer to Question 13

Method: for a parametric curve, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. The tangent is horizontal when $\frac{dy}{dt} = 0$ and vertical when $\frac{dx}{dt} = 0$.

a. Factorise $\frac{dy}{dt}$ because it will be needed in part b).

$\frac{dy}{dt} = 3t^2 - 12t + 9 = 3(t-1)(t-3)$ and $\frac{dx}{dt} = 4$, so $\frac{dy}{dx} = \frac{3(t-1)(t-3)}{4}$.

b. Set $\frac{dy}{dx} = 0$ and substitute each value of t into x and y. Horizontal when $t = 1$ and $t = 3$. $t = 1$: $(4, 4)$, a local maximum (gradient changes from $+$ to $-$). $t = 3$: $(12, 0)$, a local minimum. So the track is horizontal at $(4, 4)$ and $(12, 0)$.

c. Substitute $t = 4$ into the gradient formula. At $t = 4$: $\frac{3 \times 3 \times 1}{4} = 2.25$, so the track is climbing. So at the end the track rises 2.25 m for each metre forward.

Answer to Question 14

a. Differentiate $\cos 2t$ with the chain rule: it gives $-2 \sin 2t$. $v = -10 \sin 2t$ cm s^{-1}. The velocity is negative just after $t = 0$, so the bob first moves towards the rest position.

b. Differentiate v once more. $a = -20 \cos 2t = -4(5 \cos 2t) = -4x$. This is the equation of simple harmonic motion.

c. The bob is at rest when the velocity is zero. $v = 0$ when $\sin 2t = 0$: $2t = \pi$, so $t = \frac{\pi}{2} = 1.57$ s (at $t = 0$ the bob is also at rest, at its starting position). So the bob is first at rest again after about 1.57 seconds.

d. The sine function takes values between -1 and 1. The maximum of $|-10 \sin 2t|$ is 10 cm s^{-1}. This happens when the bob passes through its rest position.

Answer to Question 15

a. Product rule: $\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$.

b. $\frac{dy}{dx} = 0$ when $x = 0$ or $x = 2$. At $x = 0$: $(0, 0)$ is a minimum (the gradient is positive just after). At $x = 2$: $y = 4e^{-2} = 0.541$, so $(2, 0.541)$, a maximum (gradient positive before, negative after).

c. The range is $0 \leq y \leq 4e^{-2}$. As $x \to \infty$, $y \to 0$ because e^{-x} shrinks faster than x^2 grows.

d. $\frac{d^2y}{dx^2} = e^{-x}(x^2 - 4x + 2)$. At $x = 2$: $e^{-2}(4 - 8 + 2) = -2e^{-2}$, confirming a maximum.

Answer to Question 16

Method: use the gradient formula $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$. A horizontal tangent needs a zero numerator; a vertical tangent needs a zero denominator. In each case substitute back into the equation of the curve.

a. From the earlier result $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$. A horizontal tangent has $\frac{dy}{dx} = 0$, so $2x + y = 0$, i.e. $y = -2x$.

b. Substitute: $x^2 + x(-2x) + 4x^2 = 3x^2 = 7$, so $x = \pm\sqrt{\frac{7}{3}} = \pm 1.53$. Points: $(1.53, -3.06)$ and $(-1.53, 3.06)$. So the tangent is horizontal at $(1.53, -3.06)$ and $(-1.53, 3.06)$.

c. A vertical tangent has the denominator zero: $x + 2y = 0$, $x = -2y$. Substitute: $4y^2 - 2y^2 + y^2 = 3y^2 = 7$, $y = \pm 1.53$. Points: $(-3.06, 1.53)$ and $(3.06, -1.53)$. So the tangent is vertical at $(-3.06, 1.53)$ and $(3.06, -1.53)$.

Answer to Question 17

Method: for a parametric curve, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. The tangent is horizontal when $\frac{dy}{dt} = 0$ and vertical when $\frac{dx}{dt} = 0$.

a. $\frac{dx}{dt} = -3\sin t$, $\frac{dy}{dt} = 2\cos t$, so $\frac{dy}{dx} = \frac{2\cos t}{-3\sin t} = -\frac{2}{3}\cot t$.

b. At $t = \frac{\pi}{4}$: $x = \frac{3}{\sqrt{2}} = 2.12$, $y = \sqrt{2} = 1.41$ and the gradient is $-\frac{2}{3}$. Tangent: $y - \sqrt{2} = -\frac{2}{3}\left(x - \frac{3}{\sqrt{2}}\right)$, so $y = -\frac{2}{3}x + 2\sqrt{2}$.

c. $\cos^2 t + \sin^2 t = 1$ with $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$ gives $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

d. Vertical tangent when $\frac{dx}{dt} = 0$: $\sin t = 0$, $t = 0, \pi$: the points $(3, 0)$ and $(-3, 0)$.

Answer to Question 18

Method: to check a solution of a differential equation, find y' and y'', substitute into the left-hand side and show that it equals the right-hand side.

a. $y\prime = 2Ae^{2x} - 3Be^{-3x}$ and $y\prime\prime = 4Ae^{2x} + 9Be^{-3x}$. Then $y\prime\prime + y\prime - 6y = (4 + 2 - 6)Ae^{2x} + (9 - 3 - 6)Be^{-3x} = 0$.

b. $A + B = 1$ and $2A - 3B = -8$. From the first $B = 1 - A$: $2A - 3 + 3A = -8$, so $A = -1$, $B = 2$. So $y = -e^{2x} + 2e^{-3x}$.

c. $y(1) = -e^2 + 2e^{-3} = -7.39 + 0.10 = -7.29$.

d. $y \to -\infty$ because the e^{2x} term with negative coefficient dominates.

So $y = -e^{2x} + 2e^{-3x}$, $y(1) \approx -7.29$, and y decreases without limit.

Answer to Question 19

Method: in an optimisation problem, use the constraint to write the quantity in terms of

one variable, set the first derivative to zero, and use the second derivative to check the nature of the stationary point.

a. $V = \pi r^2 h = 500$ so $h = \dfrac{500}{\pi r^2}$. Then $A = \pi r^2 + 2\pi r \cdot \dfrac{500}{\pi r^2} = \pi r^2 + \dfrac{1000}{r}$.

b. $\dfrac{dA}{dr} = 2\pi r - \dfrac{1000}{r^2} = 0$ gives $r^3 = \dfrac{500}{\pi}$, so $r = \sqrt[3]{159.15} = 5.42$ cm.

c. $\dfrac{d^2A}{dr^2} = 2\pi + \dfrac{2000}{r^3} > 0$ for $r > 0$, so it is a minimum.

$A_{min} = \pi(5.42)^2 + \dfrac{1000}{5.42} = 92.3 + 184.5 = 277$ cm^2 and $h = \dfrac{500}{\pi(5.42)^2} = 5.42$ cm (equal to r).

Answer: the container uses the least material when $r = h \approx 5.42$ cm, with an area of about 277 cm².

Answer to Question 20

Method: when the variable appears in both the base and the power, take natural logarithms of both sides and differentiate implicitly.

a. $\ln y = \sin x \ln x$. Differentiate: $\dfrac{1}{y}\dfrac{dy}{dx} = \cos x \ln x + \dfrac{\sin x}{x}$. Multiply by $y = x^{\sin x}$. This uses the product rule on $\sin x \ln x$.

b. At $x = \dfrac{\pi}{2}$: $y = \dfrac{\pi}{2}$. $\cos\dfrac{\pi}{2} = 0$ and $\sin\dfrac{\pi}{2} = 1$, so the bracket is $\dfrac{1}{\pi/2} = \dfrac{2}{\pi}$. Hence $\dfrac{dy}{dx} = \dfrac{\pi}{2} \times \dfrac{2}{\pi} = 1$. So the gradient of the curve at $x = \frac{\pi}{2}$ is exactly 1.

Answer to Question 21

Method: write the tangent at a general point with parameter a, then impose the condition that it passes through the origin.

a. The tangent at (a, e^a) has gradient e^a: $y - e^a = e^a(x - a)$. It passes through $(0, 0)$: $-e^a = -ae^a$, so $a = 1$. The tangent is $y = ex$.

b. The tangent at $(a, \ln a)$ has gradient $\dfrac{1}{a}$: $y - \ln a = \dfrac{1}{a}(x - a)$. Through the origin: $-\ln a = -1$, so $a = e$. The tangent is $y = \dfrac{x}{e}$.