Test answers - Binomial expansion for rational n and series modelling

Year: Year 13 · Duration: 40 minutes · /20

Answer to Question 1 (4 marks)

Mark scheme: 1 mark for taking out the factor 2^{-3}, 1 mark for the correct coefficients -3 and 6, 1 mark for simplifying, 1 mark for the range.

$$(2+x)^{-3} = 2^{-3}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\left(1-3\cdot\frac{x}{2}+\frac{(-3)(-4)}{2}\cdot\frac{x^2}{4}\right) = \frac{1}{8}\left(1-\frac{3x}{2}+\frac{3x^2}{2}\right).$$

So the expansion is $\frac{1}{8}-\frac{3x}{16}+\frac{3x^2}{16}$ (3 marks). Valid for [redacted], i.e. $|x|<2$ (1 mark).

Answer: $\frac{1}{(2+x)^3} = \frac{1}{8}-\frac{3x}{16}+\frac{3x^2}{16}+\cdots$ for $-2<x<2$.

Check with $x=0.1$: $\frac{1}{2.1^3} = 0.10798$, while the expansion gives $0.125-0.01875+0.001875 = 0.108125$, close as expected for a small x.

Answer to Question 2 (4 marks)

Mark scheme: 2 marks for the expansion (1 for the method with $n=\frac{1}{2}$, 1 for all terms correct), 1 mark for the range, 1 mark for the estimate.

a. Use $n=\frac{1}{2}$ and inside term $u=-4x$, so $u^2=16x^2$ and $u^3=-64x^3$.
$(1+u)^{1/2} = 1+\frac{u}{2}-\frac{u^2}{8}+\frac{u^3}{16}$ with $u=-4x$: $1-2x-2x^2-4x^3+\cdots$. Every term after the first is negative.

b. The inside term is $-4x$. $|4x|<1$, so $|x|<\frac{1}{4}$. So the expansion is valid for $-\frac{1}{4}<x<\frac{1}{4}$.

c. Since $1-4(0.01)=0.96$, substitute $x=0.01$. $x=0.01$: $1-0.02-0.0002-0.000004 = 0.979796$. Calculator: 0.979796. So $\sqrt{0.96}\approx 0.979796$.

Answer to Question 3 (4 marks)

Mark scheme: 2 marks for the partial fractions (1 per constant), 1 mark for the series, 1 mark for the range.

a. Multiply both sides by $(1-x)(2+x)$ and substitute convenient values.
$4-x = A(2+x)+B(1-x)$. $x=1$: $3=3A$, $A=1$. $x=-2$: $6=3B$, $B=2$. So $h(x) = \frac{1}{1-x}+\frac{2}{2+x}$. Check at $x=0$: $1+1=2$ and $\frac{4}{2}=2$.

b. Expand each fraction with $n = -1$, taking out the factor 2 from $2 + x$. $\frac{1}{1-x} = 1 + x + x^2 + x^3$ and $\frac{2}{2+x} = \left(1 + \frac{x}{2}\right)^{-1} = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8}$. Adding: $2 + \frac{x}{2} + \frac{5}{4}x^2 + \frac{7}{8}x^3 + \cdots$. Constant term: $1 + 1 = 2$; x term: $1 - \frac{1}{2} = \frac{1}{2}$.

c. The whole series needs both conditions at the same time. The first needs $|x| < 1$ and the second $|x| < 2$, so valid for $|x| < 1$. The narrower condition wins.

Answer to Question 4 (4 marks)

Mark scheme: 1 mark for the two equations, 1 mark for a and n, 1 mark for the coefficient, 1 mark for the range.

a. Compare the general coefficients na and $\frac{n(n-1)}{2}a^2$ with the given values, then divide. $na = -3$ and $\frac{n(n-1)a^2}{2} = \frac{27}{4}$, so $(n-1)a = \frac{27/2}{-3} = -\frac{9}{2}$. Then $a = na - (n-1)a = -3 + \frac{9}{2} = \frac{3}{2}$ and $n = -2$. Check: $na = (-2) \times \frac{3}{2} = -3$.

b. Use $\frac{n(n-1)(n-2)}{3!}a^3$. $\frac{(-2)(-3)(-4)}{6}\left(\frac{3}{2}\right)^3 = -4 \times \frac{27}{8} = -\frac{27}{2}$. Valid for [redacted], so $|x| < \frac{2}{3}$. So the coefficient of x^3 is $-\frac{27}{2}$.

Answer to Question 5 (4 marks)

Mark scheme: 1 mark for the expansion, 1 mark for substituting $x = 0.04$, 1 mark for the estimate, 1 mark for the percentage.

a. Use $n = -\frac{1}{2}$ with inside term $-x$: the coefficients are $\frac{1}{2}$ and $\frac{3}{8}$. $(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3x^2}{8}$. With $x = 0.04$: $1 + 0.02 + 0.0006 = 1.0206$. Calculator: $0.96^{-1/2} = 1.020621$; the estimate is **1.02060** (to 5 d.p.). The next term, $\frac{5}{16}x^3$, would only add about 0.00002.

b. The period is multiplied by about 1.0206, so subtract 1 and convert to a percentage. The period rises by about **2.1%**. So reducing gravity by 4% makes the pendulum about 2.1% slower.