Lesson - Binomial expansion for rational n and series modelling

Year: Year 13

At AS level you expanded $(1+x)^n$ for positive integers n, where the series stops after $n+1$ terms. At A level the same formula works when n is **any rational number**, including negative numbers and fractions.

Starter activity

Use a calculator to find $\sqrt{1.1}$, $\sqrt{1.01}$ and $\sqrt{1.001}$. Subtract 1 from each answer and compare with 0.1, 0.01 and 0.001. What fraction of the small number is each result? Predict $\sqrt{1.0001}$ without a calculator, then check.

The general binomial expansion

Theorem: binomial expansion for rational n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

valid for $|x|$. When n is not a non-negative integer the series **never terminates**, so you find an approximation from the first few terms.

Watch out

Write the validity condition in every answer: it is a regular mark in exams. Each new factor in the numerator is one less than the previous one.

Worked example 1

Expand $(1+x)^{-1}$ and $\sqrt{1+x}$.

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$, the geometric series. For $n = \frac{1}{2}$ the coefficients are $\frac{1}{2}$, $\frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2} = -\frac{1}{8}$ and $\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6} = \frac{1}{16}$, so $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$ for $|x|$.

Expanding $(a+bx)^n$

Method: expanding $(a+bx)^n$

1. Factorise: $(a+bx)^n = a^n\left(1+\dfrac{b}{a}x\right)^n$, raising a to the power n.
2. Write the validity condition $\left|\dfrac{b}{a}x\right| < 1$.
3. Let $u = \dfrac{b}{a}x$, expand $(1+u)^n$, then substitute u (with brackets).
4. Multiply by a^n.

Worked example 2

Expand $\sqrt{4+x}$ up to x^2 and state the range of validity.

$2\left(1+\dfrac{x}{4}\right)^{1/2} = 2\left(1+\dfrac{x}{8}-\dfrac{x^2}{128}+\cdots\right) = 2+\dfrac{x}{4}-\dfrac{x^2}{64}+\cdots$, valid for $|x|$.

Worked example 3

Expand $(2-3x)^{-2}$ up to x^3.

$\dfrac{1}{4}\left(1-\dfrac{3x}{2}\right)^{-2}$. With $u = -\frac{3x}{2}$: $(1+u)^{-2} = 1-2u+3u^2-4u^3+\cdots$. So the expansion is $\dfrac{1}{4}+\dfrac{3}{4}x+\dfrac{27}{16}x^2+\dfrac{27}{8}x^3+\cdots$, valid for $|x|$.

Watch out

Raise the factored number to the power n: here $2^{-2} = \frac{1}{4}$, not $\frac{1}{2}$.

Using the expansion to approximate

Worked example 4

Estimate $\sqrt{1.02}$ and $\sqrt[3]{8.1}$.

For $\sqrt{1.02}$ use $x = 0.02$: $1+0.01-0.00005+0.0000005 = 1.0099505$; the calculator gives 1.00995049..., so the approximation is excellent. For $\sqrt[3]{8.1} = (8+0.1)^{1/3} = 2+\frac{x}{12}-\frac{x^2}{288}+\cdots$ with $x = 0.1$, we get 2.00830 to 5 decimal places.

Watch out

Choose x small and inside the validity range. Using $x = 3$ in $\sqrt{1+x}$ gives 3.0625 from four terms, but $\sqrt{4} = 2$: the series does not converge there.

Expanding rational functions with partial fractions

Worked example 5

Expand $\dfrac{3x+1}{(1-x)(1+2x)}$ up to x^3.

Write $\dfrac{A}{1-x}+\dfrac{B}{1+2x}$, so $3x+1=A(1+2x)+B(1-x)$. $x=1$: $A=\frac{4}{3}$. $x=-\frac{1}{2}$: $B=-\frac{1}{3}$. Expanding, $\frac{4}{3}(1+x+x^2+x^3)-\frac{1}{3}(1-2x+4x^2-8x^3)=1+2x+0x^2+4x^3+\cdots$. Valid for both $|x|$ and $|2x|$, so $|x|$.

Modelling with series

Note: series in modelling

In relativity $\gamma=\left(1-\dfrac{v^2}{c^2}\right)^{-1/2}\approx 1+\dfrac{v^2}{2c^2}+\dfrac{3v^4}{8c^4}$ for small v, so the kinetic energy $mc^2(\gamma-1)\approx\frac{1}{2}mv^2$. The pendulum period $T=2\pi\sqrt{L/g}$ increases by a factor $\sqrt{1.02}\approx 1.00995$ when the length rises by 2%, about 1%.

Worked example 6

Find the percentage error when $\sqrt{1.1}\approx 1.05$ (two terms).

The exact value is 1.0488, so the error is $\dfrac{1.05-1.0488}{1.0488}\approx 0.11\%$.

Worked example 7

A sphere's radius increases by 1%. Estimate the change in volume.

Volume is proportional to r^3, so it is multiplied by $(1.01)^3\approx 1+3(0.01)=1.03$: about a 3% rise. If the density is proportional to r^{-3} it falls by about 3%.

Method: checking an expansion

1. Choose a small value such as $x=0.1$.
2. Evaluate the original expression on a calculator.
3. Compare with the sum of your terms. If they disagree, check the signs first, then the powers of the factored number.

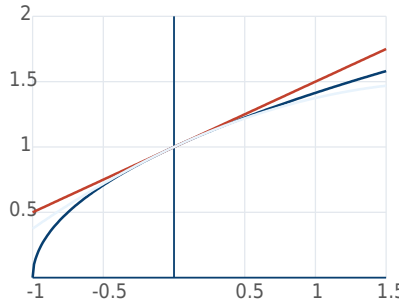
Watch out

Work in fractions, not decimals. Do not forget to factorise a first, take care with signs when b or n is negative, and write the validity as $|x|$, not $|x|$.

Why does the series stop working?

Key fact: where the series converges

The series for $(1+x)^n$ is an infinite sum. Think of $x = 0.1$: the terms $0.1, 0.01, 0.001, \ldots$ shrink rapidly, so adding more of them changes the total very little. When $|x| \geq 1$ the powers of x do not shrink, and the terms in the series never settle down. That is why the condition $|x|$ is part of the theorem. If n is a positive integer the series stops after $n+1$ terms, so it is valid for every x.



Note: reading the graph

The blue curve is $y = \sqrt{1+x}$. The straight line $y = 1 + \frac{x}{2}$ touches it at $x = 0$ and the quadratic $y = 1 + \frac{x}{2} - \frac{x^2}{8}$ hugs it more closely. Near $x = 0$ the three graphs are almost indistinguishable; further away the approximations drift apart. Each extra term widens the region where the approximation is accurate.

More worked examples

Worked example 8: negative integer powers

Expand $(1-2x)^{-2}$ up to x^3 and state when it is valid.

Use $(1+u)^{-2} = 1 - 2u + 3u^2 - 4u^3 + \cdots$ with $u = -2x$. Then $-2u = 4x$, $3u^2 = 12x^2$ and $-4u^3 = -4(-8x^3) = 32x^3$. So $(1-2x)^{-2} = 1 + 4x + 12x^2 + 32x^3 + \cdots$, valid for $|2x|$, so $|x|$.

Worked example 9: a fractional power with a factor

Expand $\dfrac{1}{\sqrt{9-x}}$ up to x^2.

Factorise:
$(9-x)^{-1/2} = 9^{-1/2}\left(1-\frac{x}{9}\right)^{-1/2} = \frac{1}{3}\left(1+\frac{x}{18}+\frac{3}{8}\cdot\frac{x^2}{81}+\cdots\right) = \frac{1}{3}+\frac{x}{54}+\frac{x^2}{648}+\cdots$
valid for $|x|$.

Worked example 10: validity with a coefficient

State the range of validity for $(3+2x)^{-1}$ and expand to x^2.

$(3+2x)^{-1} = \frac{1}{3}\left(1+\frac{2x}{3}\right)^{-1} = \frac{1}{3}\left(1-\frac{2x}{3}+\frac{4x^2}{9}-\cdots\right) = \frac{1}{3}-\frac{2x}{9}+\frac{4x^2}{27}-\cdots$. The condition is $\blacksquare$, so $|x|$. Remember to divide by the coefficient: it is $\frac{3}{2}$, not 1 and not $\frac{2}{3}$.

Worked example 11: finding a coefficient in a product

Find the coefficient of x^2 in $(1+3x)(1-x)^{-2}$.

$(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$. The x^2 terms of the product come from $1\times 3x^2$ and $3x\times 2x$. They give $3+6=9$. The coefficient is 9.

Worked example 12: a cube root estimate

Estimate $\sqrt[3]{1000.6}$ using the first two terms of a binomial expansion.

$\sqrt[3]{1000+0.6} = 10(1+0.0006)^{1/3} \approx 10\left(1+\frac{0.0006}{3}\right) = 10(1.0002) = 10.002$. The calculator gives 10.00199..., so the answer is correct to 5 significant figures.

Worked example 13: modelling a percentage change

The time for a pendulum swing is proportional to $\sqrt{L}$. Find the approximate percentage change in time when L decreases by 6%.

The time is multiplied by
$(1-0.06)^{1/2} \approx 1-0.03-\frac{0.0036}{8} = 1-0.03-0.00045 = 0.96955$. So the time falls by about 3.0%. Using only the first term gives 3%, the "rule of thumb": half the percentage change in L.

Worked example 14: partial fractions with three terms

Express $\frac{x+7}{(1+x)(1-2x)}$ as a series up to x^2.

$\frac{x+7}{(1+x)(1-2x)} = \frac{A}{1+x} + \frac{B}{1-2x}$ gives $x+7 = A(1-2x) + B(1+x)$. $x=-1$: $6 = 3A$, $A = 2$. $x = \frac{1}{2}$: $7.5 = 1.5B$, $B = 5$. Then $2(1-x+x^2) + 5(1+2x+4x^2) = 7 + 8x + 22x^2 + \cdots$, valid for $|x|$ (the more restrictive of $|x|$ and $|2x|$).

Building the coefficients step by step

Method: writing the coefficients without mistakes

1. Write the index n as a fraction or negative integer in brackets, for example $n = -\frac{1}{2}$.
2. The coefficient of u^r is the product of r factors $n,\ n-1,\ n-2, \ldots$ divided by $r!$.
3. Evaluate each product slowly, keeping signs: two negatives make a positive.
4. Only then substitute u, and remember to raise the whole of u (including its coefficient and sign) to each power.

Key fact: expansions worth memorising

These come up in almost every paper, so learn them as a set. For $|x|$:
$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$, $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$,
$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$ and $(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \cdots$.
Replace x by $-x$ to switch every sign pattern to all-positive for the first two.

Series in modelling and error analysis

Note: why scientists like small-quantity expansions

Many physical formulae contain a root or a reciprocal of an expression such as $1 + \frac{h}{R}$ or $1 - \frac{v^2}{c^2}$. When the extra quantity is tiny, the first two or three terms of the binomial series give a simple formula that is accurate to many significant figures. An engineer can then see at once how a 2% rise in length changes a period without using a calculator, and a physicist can show that relativistic energy reduces to the familiar $\frac{1}{2}mv^2$ at everyday speeds. In exam questions the comment is: "the approximation is good because x is small and inside the range of validity".

Worked example 15: percentage error

$\sqrt{1.2}$ is estimated using $1 + \frac{x}{2} - \frac{x^2}{8}$ with $x = 0.2$. Find the percentage error.

The estimate is $1 + 0.1 - 0.005 = 1.095$. The exact value is 1.095445. The error is $\frac{1.095445 - 1.095}{1.095445} \times 100 = 0.041\%$. Adding the next term $\frac{x^3}{16} = 0.0005$ gives 1.0955,

which is closer still, showing the pattern of improvement.

Worked example 16: a reciprocal in a physics formula

The resistance of a wire changes so that $R = R_0(1 + kT)^{-1}$ with $k = 0.004$. Estimate R when $T = 10$ as a fraction of R_0 using the first three terms.

$u = kT = 0.04$. $(1 + u)^{-1} \approx 1 - u + u^2 = 1 - 0.04 + 0.0016 = 0.9616$. The exact value is $\dfrac{1}{1.04} = 0.961538$, so the estimate is accurate to 3 decimal places.

Worked example 17: choosing x sensibly

Estimate $\sqrt{19}$ with two terms, first from $16\left(1 + \frac{3}{16}\right)$ and then from $20.25\left(1 - \frac{5}{81}\right)$.

First way: $4\left(1 + \dfrac{3}{32}\right) = 4.375$. Second way: $\sqrt{20.25} = 4.5$, so $4.5\left(1 - \dfrac{5}{162}\right) = 4.5 - 0.1389 = 4.3611$. The exact value is 4.3589. The second way is better because $x = \dfrac{5}{81} = 0.062$ is much smaller than $\dfrac{3}{16} = 0.19$. When estimating a root, always choose the nearest perfect square.

Note: reading examination wording

"Find the first four terms" means up to and including x^3. "Hence" means you must use the result you have just found, usually by substituting a small value of x into it. "Show that" requires every step, including the validity condition if requested. If a question says "use your expansion to estimate", leave your answer to the accuracy asked, and say whether it is an over- or under-estimate when you can.

Method: a full A level answer

1. Rewrite the bracket as $a^n(1 + u)^n$ and write down a^n.
2. State the validity: $|u|$, then solve for x.
3. Expand $(1 + u)^n$ with a clear list of coefficients.
4. Substitute for u using brackets, simplify and multiply by a^n.
5. Check by evaluating both sides at a small value of x.

Watch out: errors seen in examination scripts

- Writing the validity as $|x|$ when the bracket is $(1 + 2x)$ or $(2 + x)$.
- Treating $(2 + x)^{-1}$ as $2^{-1}(1 + x)^{-1}$: the second factor is $(1 + \frac{x}{2})^{-1}$.
- Applying the expansion with $|u| > 1$ to estimate a value, which gives nonsense.

- Losing a minus sign when u is negative, so that u^3 keeps its sign.

Did you know?

Isaac Newton discovered the general binomial series in 1665 while the plague closed Cambridge University. He tested it by finding $\sqrt{1.1}$ and other roots to many decimal places, and used it as a tool for calculus. Today the same idea is used by physicists to approximate complicated formulas when a quantity is small.

Otto's tip

Write u for the bracket term, expand $(1+u)^n$ first, and substitute u at the end. It halves the arithmetic slips.