Worksheet - Binomial expansion for rational n and series modelling

Year: Year 13 · 20

Show your working.

Question 1

Question 1 – Negative integer powers *(Section A · Skills practice)*

Expand each expression in ascending powers of x up to and including the term in x^3, and state the values of x for which each expansion is valid.

a. $(1+x)^{-2}$ [3 marks]
b. $(1-x)^{-3}$ [3 marks]
c. $(1+2x)^{-1}$ [3 marks]

Question 2

Question 2 – Fractional powers *(Section A · Skills practice)*

Find the first four terms in the expansion of each expression in ascending powers of x, and state the range of validity.

a. $(1+x)^{1/3}$ [3 marks]
b. $(1-x)^{-1/2}$ [3 marks]
c. $\sqrt{1+4x}$ [3 marks]

Question 3

Question 3 – Factorising first *(Section A · Skills practice)*

Expand each expression up to and including the term in x^2. State the range of values of x for which the expansion is valid.

a. $(4+x)^{-1/2}$ [4 marks]
b. $(2+3x)^{-2}$ [4 marks]
c. $\dfrac{1}{3-2x}$ [3 marks]

Question 4

Question 4 – Finding a coefficient *(Section A · Skills practice)*

Find the coefficient of x^3 in the expansion of each expression.

a. $(1-3x)^{-1/3}$ [3 marks]
b. $(2-x)(1+x)^{-1}$ [4 marks]
c. $\dfrac{1+x}{\sqrt{1-x}}$ [4 marks]

Question 5

Question 5 – Estimating roots *(Section A · Skills practice)*

Use the expansion of $(1+x)^{1/2}$ with the first three terms to estimate each value, giving your answer to 6 decimal places. Compare with your calculator.

a. $\sqrt{1.04}$ [3 marks]
b. $\sqrt{0.98}$ [3 marks]
c. $\sqrt{50}$, by writing $50 = 49\left(1+\frac{1}{49}\right)$ [4 marks]

Question 6

Question 6 – Validity and size of x *(Section A · Skills practice)*

A student wants to use $(1+2x)^{-1/2}$ to estimate $\dfrac{1}{\sqrt{3}}$.

a. State the range of x for which the expansion of $(1+2x)^{-1/2}$ is valid. [1 mark]
b. The student substitutes $x = 1$. Explain why this is not allowed. [1 mark]
c. Find the expansion up to x^2 and use $x = 0.01$ to estimate $\dfrac{1}{\sqrt{1.02}}$ to 5 decimal places. [4 marks]

Question 7

Question 7 – The pendulum clock *(Section B · Real-life practice)*

The period of a pendulum of length L is $T = 2\pi\sqrt{\dfrac{L}{g}}$. A grandfather clock in a hallway in Leeds is adjusted so that its pendulum has period exactly 2 seconds. In summer the brass rod expands and the length increases by 3%.

a. Show that the new period is $T_0(1.03)^{1/2}$, where T_0 is the original period. [1 mark]
b. Use the first three terms of the binomial expansion to find $(1.03)^{1/2}$ to 5 decimal places. [3 marks]
c. The clock still counts one tick per swing. Find the number of seconds the clock loses in one day, giving your answer to the nearest second. [4 marks]

Question 8

Question 8 – Speed and energy *(Section B · Real-life practice)*

In special relativity a particle of rest mass m_0 moving at speed v has kinetic energy $E = m_0c^2(\gamma - 1)$ where $\gamma = \left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$ and c is the speed of light.

a. Expand γ up to and including the term in $\dfrac{v^4}{c^4}$. [3 marks]
b. Hence show that for small v, $E \approx \frac{1}{2}m_0v^2$. [2 marks]
c. A proton travels at $v = 0.1c$. Use your expansion to estimate γ to 7 decimal places. [2 marks]

Question 9

Question 9 – Blowing bubbles *(Section B · Real-life practice)*

A bubble-blowing toy makes spherical bubbles. The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$ and its surface area is $4\pi r^2$.

a. The volume of a bubble increases by 5%. Show that the radius is multiplied by $(1.05)^{1/3}$. [2 marks]
b. Use the binomial expansion to find $(1.05)^{1/3}$ to 4 decimal places, using three terms. [3 marks]
c. Find the percentage increase in surface area, to 1 decimal place. [3 marks]

Question 10

Question 10 – The torch beam *(Section B · Real-life practice)*

The intensity of light from a lamp at distance d is $I = \dfrac{k}{d^2}$. A photographer in a studio moves a lamp 5% further from a model.

a. Show that the intensity is multiplied by $(1.05)^{-2}$. [1 mark]
b. Expand $(1 + x)^{-2}$ up to x^3 and use $x = 0.05$ to estimate $(1.05)^{-2}$ to 4 decimal places. [3 marks]
c. By what percentage does the intensity fall? [2 marks]

Question 11

Question 11 – Gravity at altitude *(Section B · Real-life practice)*

At height h above the Earth's surface the gravitational acceleration is

$g = g_0\left(1 + \dfrac{h}{R}\right)^{-2}$ where $R = 6371$ km is the radius of the Earth and $g_0 = 9.81$ m s^{-2}.

a. Use the first two terms of a binomial expansion to approximate g at the cruising height of an aeroplane, $h = 10$ km. [3 marks]
b. Find the percentage decrease in g. [2 marks]
c. Explain why the expansion is valid for the heights in this question. [1 mark]

Question 12

Question 12 – Present value of a prize *(Section B · Real-life practice)*

A school raffle promises a prize of £5000 to be paid in two years' time. The school bank account pays 3% compound interest per year. The present value is $5000(1.03)^{-2}$.

a. Use the binomial expansion of $(1 + x)^{-2}$ up to x^2 to estimate the present value. [3 marks]
b. Find the exact present value to the nearest penny. [1 mark]
c. Find the error in the estimate, in pounds. [1 mark]

Question 13

Question 13 – Wind turbine speed *(Section B · Real-life practice)*

The power output of a wind turbine is proportional to the cube of the wind speed, $P = kv^3$. Engineers want to raise the power by 8%.

a. Show that the wind speed must rise by a factor of $(1.08)^{1/3}$. [1 mark]
b. Use the first three terms of the binomial expansion to estimate this factor to 4 decimal places. [3 marks]
c. Hence estimate the percentage rise in speed to 1 decimal place. [1 mark]

Question 14

Question 14 – The builder's diagonal *(Section B · Real-life practice)*

A builder lays a square paving slab with sides 5 m. The length of the diagonal is $\sqrt{50}$ m but the builder does not have a calculator.

a. Write $\sqrt{50}$ as $7\sqrt{1 + \dfrac{1}{49}}$. [1 mark]
b. Use the binomial expansion with two terms to estimate the diagonal. [2 marks]
c. Use three terms, and find the percentage error of the two-term estimate. [3 marks]

Question 15

Question 15 – Finding unknown constants *(Section C · Problem solving)*

In the binomial expansion of $(1+ax)^n$, where a and n are constants and n is not a positive integer, the coefficients of x and x^2 are 6 and 24 respectively.

a. Show that $na = 6$ and $(n-1)a = 8$. [3 marks]
b. Find the values of a and n. [2 marks]
c. Find the coefficient of x^3. [2 marks]
d. State the range of x for which the expansion is valid. [1 mark]
e. Check your answer by substituting $x = 0.01$ into both sides. [2 marks]

Question 16

Question 16 – Partial fractions and series *(Section C · Problem solving)*

\($f(x) = \dfrac{4-5x}{(1+x)(2-x)}$\)

a. Find constants A and B such that $f(x) = \dfrac{A}{1+x} + \dfrac{B}{2-x}$. [3 marks]
b. Hence show that $f(x) = 2 - \dfrac{7}{2}x + \dfrac{11}{4}x^2 - \dfrac{25}{8}x^3 + \cdots$. [4 marks]
c. State the range of values of x for which this expansion is valid. [2 marks]
d. Use $x = 0.1$ to evaluate $f(0.1)$ exactly and by the series, and find the percentage error. [3 marks]

Question 17

Question 17 – How good is the approximation? *(Section C · Problem solving)*

A student investigates how accurately the expansion of $\sqrt{1+x}$ approximates its true value. She uses one, two and three terms after the constant 1, giving approximations A_1, A_2 and A_3.

x	$A_1 = 1 + \frac{x}{2}$	A_2 (add $-\frac{x^2}{8}$)	A_3 (add $\frac{x^3}{16}$)	Exact $\sqrt{1+x}$
0.1	1.0500	1.0488	1.0488	1.0488
0.5	1.2500	1.2188	1.2266	1.2247
1	1.5000	1.3750	1.4375	1.4142
2	2.0000	1.5000	2.0000	1.7321

a. Verify the entry for A_3 when $x = 0.5$. [2 marks]
b. Describe how the accuracy changes as terms are added when $x = 0.1$ and when $x = 0.5$. [2 marks]
c. Explain why the approximations for $x = 2$ are poor. [3 marks]

d. Find the percentage error in A_2 for $x = 0.1$. [2 marks]

Question 18

Question 18 – Relativistic mass *(Section C · Problem solving)*

The mass of a particle moving at speed v is $m = m_0\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$. A research team at a university accelerator measures the kinetic energy $E = (m - m_0)c^2$ of electrons and compares it with the classical value $\frac{1}{2}m_0v^2$.

Speed v	Fraction of c
Run A	0.05
Run B	0.20
Run C	0.50

a. Show that $E \approx \frac{1}{2}m_0v^2 + \frac{3}{8}\dfrac{m_0v^4}{c^2}$. [3 marks]

b. For Run B, find $\dfrac{E - \frac{1}{2}m_0v^2}{E}$ as a percentage using the full expression, to 2 significant figures. [4 marks]

c. For which run is the classical formula least accurate? Explain, using a calculation for Run C. [3 marks]

Question 19

Question 19 – A product expansion *(Section C · Problem solving)*

\(g(x) = \dfrac{1 + x}{\sqrt{1 - 2x}}\)

a. Expand $g(x)$ in ascending powers of x up to and including x^3. [5 marks]

b. State the range of x for which the expansion is valid. [1 mark]

c. Use $x = 0.01$ to estimate $\dfrac{1.01}{\sqrt{0.98}}$ to 6 decimal places. [2 marks]

d. The equation $g(x) = 1.05$ has a small root. Use your expansion up to x^2 to find an approximate value for it. [3 marks]

Question 20

Question 20 – The central binomial series *(Section D · Challenge)*

Consider the expansion of $(1 - 4x)^{-1/2}$.

a. Show that $(1 - 4x)^{-1/2} = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots$. [4 marks]

b. The coefficient of x^n is $\binom{2n}{n}$. Verify this for $n = 1, 2, 3, 4$. [2 marks]

c. Hence write down the first four terms of $\frac{1}{\sqrt{1-4x}}$ when $x = 0.01$ and estimate its value to 5 decimal places. [3 marks]

d. Explain why the series cannot be used to find $\frac{1}{\sqrt{1-4x}}$ when $x = \frac{1}{4}$. [1 mark]