Answers - Binomial expansion for rational n and series modelling

Year: Year 13

Answer to Question 1

Method: use $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$, replacing x by the inside term when needed, and remember that the series is only valid when the inside term has modulus less than 1.

a. Here n is a negative integer, so the series never stops. $n = -2$: coefficients -2, $\frac{(-2)(-3)}{2!} = 3$, $\frac{(-2)(-3)(-4)}{3!} = -4$. So $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$, valid for $|x|$. The coefficients alternate in sign because n is negative.

b. Writing $1 - x$ as $1 + u$ avoids sign errors. Let $u = -x$ and use $(1+u)^{-3} = 1 - 3u + 6u^2 - 10u^3 + \cdots$ (coefficients -3, $\frac{(-3)(-4)}{2} = 6$, $\frac{(-3)(-4)(-5)}{6} = -10$). Then $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$, valid for $|x|$. Every coefficient is now positive, since $u = -x$ changes each odd power.

c. This is a geometric series with ratio $-2x$. Let $u = 2x$: $(1+u)^{-1} = 1 - u + u^2 - u^3 + \cdots$. So $(1+2x)^{-1} = 1 - 2x + 4x^2 - 8x^3 + \cdots$. It is valid when $|2x|$, that is $|x|$. The range of validity is narrower than in parts a) and b).

Answer to Question 2

Method: with a fractional power the expansion is infinite. Work out each coefficient carefully as a product of fractions, then substitute the inside term and simplify.

a. Take $n = \frac{1}{3}$. Coefficients: $\frac{1}{3}$, $\frac{\frac{1}{3}\cdot(-\frac{2}{3})}{2} = -\frac{1}{9}$, $\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{6} = \frac{5}{81}$. So $(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \cdots$, valid for $|x|$. So the first four terms are 1, $\frac{x}{3}$, $-\frac{x^2}{9}$ and $\frac{5x^3}{81}$.

b. Take $n = -\frac{1}{2}$ and replace u by $-x$. $(1+u)^{-1/2} = 1 - \frac{u}{2} + \frac{3u^2}{8} - \frac{5u^3}{16} + \cdots$ with $u = -x$: $(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \cdots$, valid for $|x|$. All the terms are positive because the minus signs cancel in pairs.

c. Write $\sqrt{1+4x} = (1+4x)^{1/2}$. $(1+u)^{1/2} = 1 + \frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{16} - \cdots$ with $u = 4x$: $1 + 2x - 2x^2 + 4x^3 - \cdots$. Valid when $|4x|$, so $|x|$. Notice that each power of $4x$ must be calculated in full: $(4x)^2 = 16x^2$ and $(4x)^3 = 64x^3$.

Answer to Question 3

Method: the binomial series needs the form $(1 + \text{something})^n$. So first take out the constant term as a factor, raised to the power n, then expand and multiply back.

a. Take out 4 from the bracket, which gives the factor $4^{-1/2} = \frac{1}{2}$.
$(4+x)^{-1/2} = 4^{-1/2}\left(1 + \frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(1 - \frac{x}{8} + \frac{3}{8}\cdot\frac{x^2}{16} - \cdots\right) = \frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256} - \cdots$
. Valid for , so $|x|$. So the expansion is $\frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256}$, valid for -4.

b. Take out 2, which gives the factor $2^{-2} = \frac{1}{4}$.
$(2+3x)^{-2} = \frac{1}{4}\left(1 + \frac{3x}{2}\right)^{-2} = \frac{1}{4}\left(1 - 3x + \frac{27x^2}{4} - \cdots\right) = \frac{1}{4} - \frac{3x}{4} + \frac{27x^2}{16} - \cdots$
using $(1+u)^{-2} = 1 - 2u + 3u^2$ with $u = \frac{3x}{2}$. Valid for $|x|$. The validity comes from .

c. Write the fraction as $(3-2x)^{-1}$ and take out 3.
$(3-2x)^{-1} = \frac{1}{3}\left(1 - \frac{2x}{3}\right)^{-1} = \frac{1}{3}\left(1 + \frac{2x}{3} + \frac{4x^2}{9} + \cdots\right) = \frac{1}{3} + \frac{2x}{9} + \frac{4x^2}{27} + \cdots$
. Valid for $|x|$. The validity comes from , which gives $|x|$.

Answer to Question 4

Method: you do not need the whole expansion. Find only the terms that can produce x^3, and when a bracket multiplies a series, collect every product whose powers add up to 3.

a. The x^3 term is $\frac{n(n-1)(n-2)}{3!}(-3x)^3$ with $n = -\frac{1}{3}$. Coefficient $= \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!}(-3)^3 = \frac{-28/27}{6} \times (-27) = \frac{28}{6} = \frac{14}{3}$. The coefficient of x^3 is $\frac{14}{3}$.

b. Expand $(1+x)^{-1}$ first, then multiply by each term of $2 - x$.
$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$. Multiplying by $(2-x)$, the x^3 terms are $2(-x^3) + (-x)(x^2) = -2x^3 - x^3 = -3x^3$. Coefficient $= -3$. The two contributions come from $2 \times (-x^3)$ and $(-x) \times x^2$.

c. Write the expression as $(1+x)(1-x)^{-1/2}$.

$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \cdots$. Multiplying by $(1+x)$: x^3 terms are $\frac{5}{16} + \frac{3}{8} = \frac{11}{16}$. The x^3 terms come from $1 \times \frac{5}{16}x^3$ and $x \times \frac{3}{8}x^2$, so the coefficient is $\frac{11}{16}$.

Answer to Question 5

Method: use $(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$. Choose x so that $1+x$ is the number under the square root, and check that $|x|$ so the series converges.

a. Here $1.04 = 1 + 0.04$. $x = 0.04$: $1 + 0.02 - \frac{0.0016}{8} = 1 + 0.02 - 0.0002 = 1.0198$. Calculator: 1.019804. The error is about 4×10^{-6}. The estimate is very accurate because x is small.

b. Here $0.98 = 1 - 0.02$, so x is negative. $x = -0.02$: $1 - 0.01 - \frac{0.0004}{8} = 1 - 0.01 - 0.00005 = 0.98995$. Calculator: 0.989949. The error is about 5×10^{-7}. The estimate is even better because $|x|$ is smaller than in part a).

c. We cannot use $x = 49$ because the series would diverge, so we factor out the perfect square 49. $\sqrt{50} = 7\sqrt{1 + \frac{1}{49}} \approx 7\left(1 + \frac{1}{98} - \frac{1}{8 \times 2401}\right) = 7 \times 1.0101523 = 7.071066$. Calculator: 7.071068, so the estimate is correct to 5 decimal places. This shows why factoring out the nearest perfect square is such a powerful trick.

Answer to Question 6

Method: the expansion of $(1+2x)^n$ is only valid when the inside term $2x$ has modulus less than 1. Always check this before substituting a value.

a. The condition for validity is that the term added to 1 has modulus less than 1. $|2x|$ so $|x|$. So the expansion is valid for $-\frac{1}{2}$.

b. Check the condition with the chosen value. $x = 1$ is outside the range $|x|$, so the series does not converge to $\frac{1}{\sqrt{3}}$ (the terms $1 - x + \frac{3}{2}x^2 - \cdots$ get larger, not smaller). So substituting $x = 1$ gives a meaningless answer.

c. Find the coefficients with $n = -\frac{1}{2}$ and inside term $2x$. $(1+2x)^{-1/2} = 1 - x + \frac{3}{2}x^2 - \cdots$ (using $-\frac{1}{2}$, then $\frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(2x)^2 = \frac{3}{2}x^2$). With $x = 0.01$: $1 - 0.01 + 0.00015 = 0.99015$. Calculator: 0.990148, so 0.99015 to 5 d.p. The value $x = 0.01$ is inside the range of validity, so the estimate can be trusted.

Answer to Question 7

a. Replace L by $1.03L_0$ in the formula and separate the square root.
$T = 2\pi\sqrt{\dfrac{1.03L_0}{g}} = \sqrt{1.03} \times 2\pi\sqrt{\dfrac{L_0}{g}} = T_0(1.03)^{1/2}$. This proves the result.

b. Use $(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$ with $x = 0.03$. $x = 0.03$:
$1 + \dfrac{0.03}{2} - \dfrac{0.0009}{8} = 1 + 0.015 - 0.0001125 = 1.0148875$. So $(1.03)^{1/2} \approx$ **1.01489** (calculator 1.014889). The three-term estimate agrees with the calculator to 5 decimal places.

c. A longer period means each tick takes longer, so the clock runs slow. Each swing now takes $2 \times 1.0148875 = 2.029775$ s. In one real day of 86 400 s the clock shows $86400 \div 1.0148875 = 85\,134$ s. It loses $86400 - 85134 \approx$ **1267 s** (about 21 minutes). So the clock loses about 1267 seconds per day, which is roughly 21 minutes.

Answer to Question 8

a. Write $\gamma = (1-u)^{-1/2}$ with $u = \frac{v^2}{c^2}$ and use $n = -\frac{1}{2}$. Let $u = \dfrac{v^2}{c^2}$.
$(1-u)^{-1/2} = 1 + \dfrac{u}{2} + \dfrac{3u^2}{8} + \cdots$, so $\gamma \approx 1 + \dfrac{v^2}{2c^2} + \dfrac{3v^4}{8c^4}$. This expansion is valid because v, so $0 \le u$.

b. Substitute the expansion of γ into $E = m_0c^2(\gamma - 1)$.
$E = m_0c^2(\gamma - 1) \approx m_0c^2\left(\dfrac{v^2}{2c^2} + \cdots\right) = \frac{1}{2}m_0v^2 + \dfrac{3m_0v^4}{8c^2}$. For small v the second term is tiny, so $E \approx \frac{1}{2}m_0v^2$, the classical kinetic energy. This shows that Newton's formula is the first term of the relativistic formula.

c. With $v = 0.1c$, $u = 0.1^2 = 0.01$. $u = 0.01$:
$\gamma \approx 1 + 0.005 + \dfrac{3(0.0001)}{8} = 1 + 0.005 + 0.0000375 =$ **1.0050375** (exact 1.0050378). The estimate differs from the exact value only in the seventh decimal place.

Answer to Question 9

a. Write the new volume as 1.05 times the old one and cancel $\frac{4}{3}\pi$.
$\dfrac{4}{3}\pi r_1^3 = 1.05 \times \dfrac{4}{3}\pi r_0^3$ gives $r_1 = r_0(1.05)^{1/3}$. Taking the cube root of both sides proves the result.

b. Use $n = \frac{1}{3}$ and $x = 0.05$. $(1+x)^{1/3} = 1 + \dfrac{x}{3} - \dfrac{x^2}{9}$; with $x = 0.05$:

$1 + 0.016667 - 0.000278 = 1.016389$, so **1.0164** to 4 d.p. So the radius increases by about 1.64%.

c. Surface area depends on r^2, so it is multiplied by the square of the radius factor. Area is multiplied by $(1.05)^{2/3} = (1.016389)^2 = 1.03305$. The surface area increases by about **3.3%**. So the surface area rises by about 3.3%, less than the 5% rise in volume.

Answer to Question 10

a. Replace d by $1.05d$ in the formula. $I = \dfrac{k}{(1.05d)^2} = \dfrac{k}{d^2}(1.05)^{-2}$. So the intensity is multiplied by $(1.05)^{-2}$.

b. Use $n = -2$ and $x = 0.05$. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3$: $1 - 0.1 + 0.0075 - 0.0005 = 0.907$. Calculator: 0.907029, so **0.9070** to 4 d.p. Four terms already give 4 decimal places of accuracy.

c. Subtract the multiplier from 1: $1 - 0.907 = 0.093$. The intensity becomes 0.907 of its original value, a fall of about **9.3%**. So moving the lamp only 5% further away reduces the light by almost twice that percentage.

Answer to Question 11

a. Let $x = \frac{h}{R}$ and use $(1+x)^{-2} \approx 1 - 2x$. $x = \dfrac{10}{6371} = 0.0015696$. $g \approx 9.81(1 - 2x) = 9.81 \times 0.996861 =$ **9.779** m s^{-2}. So at cruising height g is about 9.779 m s^{-2}.

b. The relative change is $-2x$, so convert $2x$ into a percentage. $2x = 0.31392\%$, so the decrease is about **0.31%**. So gravity is only about 0.31% weaker at 10 km.

c. State the condition for the series and compare it with the given height. The expansion needs $|$, that is h km. Since 10 km is far smaller, the series converges very quickly. So the two-term approximation is both valid and accurate.

Answer to Question 12

a. Use $(1+x)^{-2} \approx 1 - 2x + 3x^2$ with $x = 0.03$, then multiply by 5000. $x = 0.03$: $(1.03)^{-2} \approx 1 - 0.06 + 3(0.0009) = 0.9427$. Present value $\approx 5000 \times 0.9427 =$ **£4713.50**. So the estimated present value is £4713.50.

b. Compute $(1.03)^2 = 1.0609$ and divide. $5000 \div 1.0609 =$ **£4712.98**. This is the exact present value.

c. Subtract the exact value from the estimate. $4713.50 - 4712.98 =$ **£0.52**. So the binomial estimate is only 52p too high, an error of about 0.01%.

Answer to Question 13

a. Let the speed be multiplied by a and compare the new power with 1.08 times the old power. $k(av)^3 = 1.08kv^3$ gives $a^3 = 1.08$, so $a = (1.08)^{1/3}$. So the speed must rise by a factor of $(1.08)^{1/3}$.

b. Use $(1+x)^{1/3} \approx 1 + \frac{x}{3} - \frac{x^2}{9}$ with $x = 0.08$. $x = 0.08$: $1 + \dfrac{0.08}{3} - \dfrac{0.0064}{9} = 1 + 0.026667 - 0.000711 = 1.025956$. So **1.0260** to 4 d.p. (calculator 1.025986). The three-term estimate is about 0.003% below the true value.

c. Convert the factor into a percentage increase. The speed rises by about **2.6%**. So an 8% rise in power needs only about a 2.6% rise in wind speed.

Answer to Question 14

a. Factor out the perfect square 49 inside the root. $7^2 = 49$, so $\sqrt{50} = \sqrt{49 \times \frac{50}{49}} = 7\sqrt{1 + \frac{1}{49}}$. This puts the root in the form $7(1+x)^{1/2}$ with $x = \frac{1}{49}$.

b. Use the first two terms $1 + \frac{x}{2}$ and multiply by 7. $7\left(1 + \dfrac{1}{98}\right) = 7 + \dfrac{1}{14} = 7.0714$ m. So the two-term estimate of the diagonal is about 7.0714 m.

c. Add the third term $-\frac{x^2}{8}$ and compare both estimates with the exact value. Three terms: $7\left(1 + \dfrac{1}{98} - \dfrac{1}{19208}\right) = 7.0710641$. Exact $\sqrt{50} = 7.0710678$. Two-term error: $\dfrac{7.0714286 - 7.0710678}{7.0710678} = 0.0051\%$. So the two-term estimate is already within about 0.005% of the true length.

Answer to Question 15

a. Compare the general coefficients na and $\frac{n(n-1)}{2}a^2$ with 6 and 24. Coefficient of x: $na = 6$. Coefficient of x^2: $\dfrac{n(n-1)}{2}a^2 = 24$, so $n(n-1)a^2 = 48$. Substituting $na = 6$: $6(n-1)a = 48$, so $(n-1)a = 8$. These are the two required equations.

b. Subtract the two linear equations to eliminate n. Subtracting: $na - (n-1)a = a = 6 - 8 = -2$. Then $n = \dfrac{6}{a} = -3$. So $a = -2$, $n = -3$. Check: $na = (-3)(-2) = 6$ and $(n-1)a = (-4)(-2) = 8$.

c. Use $\frac{n(n-1)(n-2)}{3!}a^3$. $\dfrac{(-3)(-4)(-5)}{6}(-2)^3 = (-10)(-8) = \mathbf{80}$. So the coefficient of x^3 is 80.

d. The inside term is $-2x$. $|-2x|$, so $|x|$. So the expansion is valid for $-\frac{1}{2}$.
e. Evaluate the exact value and the four-term series. LHS $= (1-0.02)^{-3} = 1.06248\ldots$ and RHS $= 1 + 0.06 + 0.0024 + 0.00008 = 1.06248$. They agree to 5 decimal places. The agreement confirms the values of a and n.

Answer to Question 16

a. Multiply both sides by the denominator and substitute values of x that remove one constant each time. $4 - 5x = A(2-x) + B(1+x)$. Put $x = -1$: $9 = 3A$ so $A = 3$. Put $x = 2$: $-6 = 3B$ so $B = -2$. So $f(x) = \dfrac{3}{1+x} - \dfrac{2}{2-x}$.
b. Expand each fraction separately with $n = -1$, taking out the factor 2 in the second one. $\dfrac{3}{1+x} = 3(1 - x + x^2 - x^3 + \cdots)$.
$\dfrac{-2}{2-x} = -\left(1 - \dfrac{x}{2}\right)^{-1} = -\left(1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{8} + \cdots\right)$. Adding: constant $3 - 1 = 2$; x: $-3 - \dfrac{1}{2} = -\dfrac{7}{2}$; x^2: $3 - \dfrac{1}{4} = \dfrac{11}{4}$; x^3: $-3 - \dfrac{1}{8} = -\dfrac{25}{8}$. Collecting like terms gives the required series.
c. Each fraction has its own range of validity; the whole series needs both. The first part needs $|x|$ and the second needs , i.e. $|x|$. Both hold when $|x|$ **1**. So the expansion is valid for -1.
d. Substitute $x = 0.1$ into the fraction and into four terms of the series. Exact: $f(0.1) = \dfrac{3.5}{1.1 \times 1.9} = 1.674641$. Series: $2 - 0.35 + 0.0275 - 0.003125 = 1.674375$. Error: $\dfrac{0.000266}{1.674641} \approx 0.016\%$. So the percentage error is only about 0.016%.

Answer to Question 17

a. $A_3 = 1 + 0.25 - \dfrac{0.25}{8} + \dfrac{0.125}{16} = 1 + 0.25 - 0.03125 + 0.0078125 = 1.2266$. The exact value is $\sqrt{1.5} = 1.2247$.
b. For $x = 0.1$ the values 1.0500, 1.0488, 1.0488 move quickly to the exact value 1.0488. For $x = 0.5$ the values 1.2500, 1.2188, 1.2266 swing either side of 1.2247 and close in more slowly, so more terms are needed for the same accuracy.
c. For $x = 2$ the values are 2.0000, 1.5000, 2.0000 while the exact value is $\sqrt{3} = 1.7321$. The sequence does not settle. The expansion of $(1+x)^{1/2}$ is only valid for $|x|$ and $x = 2$ is outside this range.
d. $A_2 = 1 + 0.05 - 0.00125 = 1.04875$. Exact $= 1.048809$. Error $= \dfrac{1.048809 - 1.04875}{1.048809} \approx 0.0056\%$.

Answer to Question 18

a. $\left(1-\frac{v^2}{c^2}\right)^{-1/2} \approx 1+\frac{v^2}{2c^2}+\frac{3v^4}{8c^4}$, so $E = m_0c^2(\gamma-1) \approx \frac{1}{2}m_0v^2+\frac{3m_0v^4}{8c^2}$.

b. $v = 0.2c$: $\gamma = (1-0.04)^{-1/2} = 1.0206207$, so $E = 0.0206207\,m_0c^2$. Classical value $= \frac{1}{2}(0.04) = 0.02\,m_0c^2$. Difference $= 0.0006207$, which is $\frac{0.0006207}{0.0206207} =$ **3.0%** of E.

c. Run C is least accurate. $v = 0.5c$: $\gamma = (0.75)^{-1/2} = 1.154701$, $E = 0.154701\,m_0c^2$ while the classical value is $0.125\,m_0c^2$, so the classical formula is 19% too low. The expansion needs $\frac{v^2}{c^2}$ small, and 0.25 is not small.

Answer to Question 19

a. Expand $(1-2x)^{-1/2}$ with inside term $-2x$, then multiply by $(1+x)$ and collect terms.
$(1-2x)^{-1/2} = 1+\frac{1}{2}(2x)+\frac{3}{8}(2x)^2+\frac{5}{16}(2x)^3+\cdots = 1+x+\frac{3}{2}x^2+\frac{5}{2}x^3+\cdots$.
Multiplying by $(1+x)$:
$1+(1+1)x+\left(\frac{3}{2}+1\right)x^2+\left(\frac{5}{2}+\frac{3}{2}\right)x^3 = 1+2x+\frac{5}{2}x^2+4x^3+\cdots$. So $g(x) = 1+2x+\frac{5}{2}x^2+4x^3+\cdots$.

b. The validity only depends on the bracket with a non-integer power. $|2x|$, so $|x|$. So the expansion is valid for $-\frac{1}{2}$.

c. Substitute $x = 0.01$ into the four terms. $1+0.02+0.00025+0.000004 = 1.020254$. Calculator: 1.020254. So the estimate is 1.020254, correct to 6 decimal places.

d. Replace $g(x)$ by its quadratic approximation and solve with the formula.
$1+2x+\frac{5}{2}x^2 = 1.05$ gives $5x^2+4x-0.1 = 0$, so
$x = \frac{-4+\sqrt{16+2}}{10} = \frac{-4+4.2426}{10} =$ **0.0243** (taking the small positive root). So the small root is approximately $x = 0.0243$.

Answer to Question 20

a. $(1+u)^{-1/2} = 1-\frac{u}{2}+\frac{3}{8}u^2-\frac{5}{16}u^3+\frac{35}{128}u^4-\cdots$. With $u = -4x$:
$1+2x+\frac{3}{8}(16x^2)+\frac{5}{16}(64x^3)+\frac{35}{128}(256x^4) = 1+2x+6x^2+20x^3+70x^4+\cdots$.

b. $\binom{2}{1} = 2$, $\binom{4}{2} = 6$, $\binom{6}{3} = 20$, $\binom{8}{4} = 70$. All match.

c. $x = 0.01$: $1+0.02+0.0006+0.00002 = 1.02062$ (the next term is 0.0000007). Calculator: $0.96^{-1/2} = 1.020621$, so 1.02062 to 5 d.p.

d. $x = \frac{1}{4}$ gives $|4x| = 1$, which is not less than 1, so the series (with coefficients $\binom{2n}{n} \to \infty$) diverges; the function itself is undefined at $x = \frac{1}{4}$.