Lesson - Vectors in two dimensions

Year: Year 12

A **vector** has both magnitude (size) and direction. Vectors describe displacement, velocity and force. You will use them to solve geometry problems and to prove results about shapes.

Starter activity

You are in a park and walk 40 paces east and then 30 paces north. A friend walks directly to the same spot in a straight line. How many paces does your friend walk, and in which direction compared with north? Now your friend walks the same route twice as fast as you. Where is your friend when you have walked for one minute? Think about how to describe each journey with two numbers, and how a single arrow could replace the two steps.

Notation and operations

Definition: vector notation

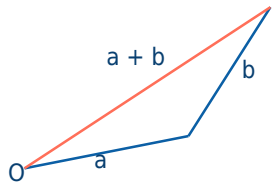
Write a vector as a column $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ or in unit-vector form $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions. By hand, underline vectors: $\underline{a}$. The vector from A to B is $\overrightarrow{AB}$.

Rule: adding and scaling vectors

- Addition: add components. $\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
- Scalar multiple: multiply each component. $2\mathbf{a} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$.
- Subtraction: $\mathbf{a} - \mathbf{b}$ means add $-\mathbf{b}$, which points the opposite way.

Note: triangle law

$\mathbf{a}$ followed by $\mathbf{b}$ gives the **resultant** $\mathbf{a} + \mathbf{b}$.



Magnitude and direction

Definition: magnitude and unit vector

The magnitude of $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$, by Pythagoras. For example [redacted]. A **unit vector** has magnitude 1; the unit vector in the direction of $\mathbf{a}$ is $\dfrac{\mathbf{a}}{|\mathbf{a}|}$.

Key fact: direction angle

The angle θ between $\begin{pmatrix} x \\ y \end{pmatrix}$ and the positive x-axis satisfies $\tan\theta = \dfrac{y}{x}$. Check the quadrant with a sketch.

Worked example 1

Find the unit vector in the direction of $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

Magnitude $= 10$, so the unit vector is $\begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$. A vector of magnitude 5 in the same direction is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Worked example 2

The vector $\begin{pmatrix} \lambda \\ 2 \end{pmatrix}$ has magnitude $\sqrt{13}$. Find λ.

$\lambda^2 + 4 = 13$, so $\lambda = \pm 3$: the vector can point either side of the y-axis.

Parallel vectors

Definition: parallel and collinear

Two vectors are **parallel** if one is a scalar multiple of the other: $\mathbf{b} = \lambda\mathbf{a}$. Three points

A, B, C are **collinear** if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel (they share the point B).

Worked example 3

Show that $A(1,1)$, $B(3,2)$ and $C(7,4)$ are collinear.

$\overrightarrow{AB}$ and $\overrightarrow{BC}$. The vectors are parallel and share the point B, so the three points lie on one straight line.

Worked example 4

Find s and t so that $s\begin{pmatrix}2\\1\end{pmatrix}+t\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$.

Compare components: $2s-t=3$ and $s+3t=5$. With $t=2s-3$ we get $7s=14$, so $s=2$ and $t=1$.

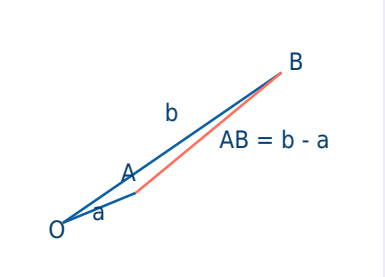
Position vectors

Definition: position vector

The **position vector** of A is $\overrightarrow{OA}$, the vector from the origin O.

Rule: vector between points, midpoint and ratio

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$, because $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$. The midpoint M of AB has $\overrightarrow{OM}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})$. A point dividing AB in the ratio $m:n$ has position vector $\mathbf{a}+\frac{m}{m+n}(\mathbf{b}-\mathbf{a})$.



Worked example 5

$A(1,2)$ and $B(7,8)$. The point P divides AB in the ratio $1:2$. Find .

, so .

Geometric proofs

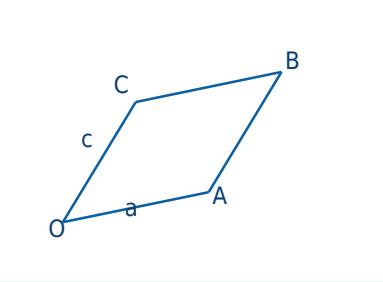
Method: vector proofs

1. Draw a diagram and name two vectors, for example $\mathbf{a}$ and $\mathbf{c}$.
2. Express every side and diagonal in terms of them.
3. Compare: parallel means a scalar multiple; "same point" means equal position vectors.
4. Finish with a statement in words.

Worked example 6

Prove that the diagonals of a parallelogram $OABC$ bisect each other.

and , so and . The midpoint of OB is $\frac{1}{2}(\mathbf{a}+\mathbf{c})$, and the midpoint of AC is $\mathbf{a}+\frac{1}{2}(\mathbf{c}-\mathbf{a})=\frac{1}{2}(\mathbf{a}+\mathbf{c})$. They are the same point, so the diagonals bisect each other. ■



Velocity and bearings

Rule: speed and bearing

With $\mathbf{i}$ pointing east and $\mathbf{j}$ north, a speed v on bearing θ (clockwise from north) has components $\begin{pmatrix} v\sin\theta \\ v\cos\theta \end{pmatrix}$. Velocity $3\mathbf{i}+4\mathbf{j}$ km/h has speed 5 km/h and bearing $\tan^{-1}\frac{3}{4}=36.9°$.

Watch out

, not $\mathbf{a} - \mathbf{b}$. Do not add magnitudes instead of components. Divide by the magnitude to make a unit vector. Check the quadrant for direction angles. Collinearity needs a common point, not just parallel vectors.

Worked example 7: A unit vector and a vector of given length

Find a vector of magnitude 10 in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

$|\mathbf{v}| = \sqrt{9 + 16} = 5$. The unit vector is $\frac{1}{5}(3\mathbf{i} - 4\mathbf{j})$. Multiply by 10: $2(3\mathbf{i} - 4\mathbf{j}) = 6\mathbf{i} - 8\mathbf{j}$, whose magnitude is $\sqrt{36 + 64} = 10$ ✓. To get a vector of magnitude m in the direction of $\mathbf{v}$, multiply the unit vector $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ by m. To get the opposite direction use $-m\hat{\mathbf{v}}$.

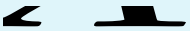
Worked example 8: Finding the angle between a vector and the axes

Find the angle that $\mathbf{u} = -2\mathbf{i} + 7\mathbf{j}$ makes with the positive y-axis.

Draw a sketch: $\mathbf{u}$ points up and to the left. The angle with the positive y-axis is $\tan^{-1}\left(\frac{2}{7}\right) = 15.9°$ (to the left of the axis). Angle with the positive x-axis is $180° - \tan^{-1}\left(\frac{7}{2}\right) = 180° - 74.1° = 105.9°$. Check: $105.9° - 90° = 15.9°$ ✓. Always sketch the vector first so that you place the angle in the right quadrant.

Worked example 9: Three points in a straight line

$A(2, 1)$, $B(5, 7)$ and $C(k, 13)$ are collinear. Find k and the ratio $AB : BC$.

 and . For collinear points the vectors must be parallel: . The $\mathbf{j}$ components give $6 = 6\lambda$, so $\lambda = 1$ and then $k - 5 = 3$, so $k = 8$. The ratio is $AB : BC = 1 : 1$ so B is the midpoint of AC. Check the midpoint: $\left(\frac{2+8}{2}, \frac{1+13}{2}\right) = (5, 7)$ ✓.

Worked example 10: Resolving a velocity from a bearing

A cyclist rides at 12 m/s on a bearing of $240°$. Find the velocity as $a\mathbf{i} + b\mathbf{j}$ and the time to travel 100 m west of the start.

East component $= 12\sin 240° = -10.39$ and north component $= 12\cos 240° = -6$. So $\mathbf{v} = -10.39\mathbf{i} - 6\mathbf{j}$: the cyclist moves west and south. To be 100 m west of the start we need $-10.39t = -100$, so $t = 9.62$ s, by which time the cyclist has also gone $6 \times 9.62 = 57.7$ m south. The bearing $240°$ is measured clockwise from north, so the angle to the west-south direction is $30°$ south of west.

Worked example 11: Where do two moving objects meet?

A particle starts at $(1, 2)$ with velocity $2\mathbf{i} + \mathbf{j}$. Another starts at $(9, -2)$ with velocity $-2\mathbf{i} + 3\mathbf{j}$. Do they collide?

Positions at time t: $(1 + 2t,\ 2 + t)$ and $(9 - 2t,\ -2 + 3t)$. Equal x: $1 + 2t = 9 - 2t$ gives $t = 2$. Equal y: $2 + t = -2 + 3t$ gives $t = 2$. Both conditions give the same time, so the particles collide at $t = 2$ at the point $(5, 4)$. Always solve the $\mathbf{i}$ and $\mathbf{j}$ equations separately and check that the times agree.

Worked example 12: Ratio and position vectors

and . P lies on AB with $AP : PB = 3 : 1$. Show that

.

. Since $AP : PB = 3 : 1$, $AP = \frac{3}{4}AB$. So

. A useful pattern: if $AP : PB = m : n$ then . Check with the midpoint ($m = n = 1$): $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ ✓. The weights on $\mathbf{a}$ and $\mathbf{b}$ add to 1.

Worked example 13: Identifying a quadrilateral from its vertices

$P(0, 1)$, $Q(4, 3)$, $R(7, 1)$ and $S(3, -1)$. Show that $PQRS$ is a parallelogram and decide whether it is a rhombus or a rectangle.

and , so PQ and SR are equal and parallel: $PQRS$ is a parallelogram. , so $|PQ| = \sqrt{20}$ and $|QR| = \sqrt{13}$. The adjacent sides are different lengths, so it is not a rhombus. For a rectangle we need $PQ \perp QR$, which needs the dot product, or the gradients $\frac{1}{2}$ and $-\frac{2}{3}$ to multiply to -1; $\frac{1}{2} \times \left(-\frac{2}{3}\right) = -\frac{1}{3}$, so it is not a rectangle either. It is a general parallelogram.

Worked example 14: Distance between two moving objects

At time t seconds a toy car is at $(t, 2t)$ and a second toy car is at $(6 - t, 1 + t)$ in metres. Find when they are closest and the least distance.

The vector between them is $(6 - 2t)\mathbf{i} + (1 - t)\mathbf{j}$. The squared distance is $(6 - 2t)^2 + (1 - t)^2 = 36 - 24t + 4t^2 + 1 - 2t + t^2 = 5t^2 - 26t + 37$. Complete the square: $5\left(t - \frac{13}{5}\right)^2 + 37 - \frac{169}{5} = 5\left(t - \frac{13}{5}\right)^2 + \frac{16}{5}$. The least squared distance is $\frac{16}{5} = 3.2$ when $t = 2.6$ s, so the least distance is $\sqrt{3.2} = 1.79$ m. This method works for any two objects moving at constant velocity, because the squared distance is always a

quadratic in t.

Method: a vector geometry proof

1. Label the position vectors or the vectors along two sides, for example  and .
2. Write each required vector by a journey along known vectors; going backwards gives the negative.
3. Simplify to the form $x\mathbf{a} + y\mathbf{b}$.
4. To prove parallel lines show one vector is a multiple of the other; to prove collinear points also say the vectors share a common point.
5. State the conclusion in words.

Watch out: common vector mistakes

- Writing . It is $\mathbf{b} - \mathbf{a}$ (end minus start).
- Mixing up bearings and angles from the x-axis: a bearing is measured clockwise from north.
- Adding magnitudes instead of vectors: $|\mathbf{a} + \mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$ unless they are parallel.
- Assuming two lines are parallel without comparing the ratios of both components.
- Writing a final answer as a coordinate pair when the question asks for a position vector or vice versa.

Note: vectors and forces

A vector keeps its meaning wherever it is drawn, so the same rules work for displacements, velocities, accelerations and forces. In mechanics you will use $\mathbf{F} = m\mathbf{a}$ with column vectors, and the resultant of several forces is found by adding vectors. That is why the work on resolving a velocity into east and north components here is the basis of the mechanics chapters later in the course.

Did you know?

Vectors were developed in the 1880s by Josiah Willard Gibbs and Oliver Heaviside, who found that the long expressions used by physicists to describe electric and magnetic forces could be shortened by treating a quantity with both size and direction as a single object. Heaviside called it a "vector", from the Latin for "carrier". Today air-traffic controllers, video-game designers and sailors all add vectors constantly, as in the wind and river examples in this chapter. Whenever a question describes

movement with a speed and a direction, ask yourself which components are east and which are north, then add the vectors and take the magnitude of the result to get the final speed or distance.

Otto's tip

To go from A to B, "go back to the origin, then forward": $-\mathbf{a}+\mathbf{b}$.