Worksheet - Vectors in two dimensions

Year: Year 12 · 21

Show your working.

Question 1

Question 1 – Vector arithmetic in i, j form *(Section A · Skills practice)*

$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + 5\mathbf{j}$.

a. Find $\mathbf{a} + \mathbf{b}$. [1 mark]
b. Find $2\mathbf{a} - 3\mathbf{b}$. [2 marks]
c. Find $|\mathbf{a}|$ as a surd. [2 marks]
d. Find a unit vector in the direction of $\mathbf{b}$. [2 marks]

Question 2

Question 2 – Magnitude and direction *(Section A · Skills practice)*

Give angles to 1 decimal place.

a. Find the magnitude of $-5\mathbf{i} + 12\mathbf{j}$ and the angle it makes with the positive x-axis (measured anticlockwise). [4 marks]
b. Find the magnitude of $4\mathbf{i} - 3\mathbf{j}$ and its direction as an angle between $0°$ and $360°$. [4 marks]
c. Write, in the form $p\mathbf{i} + q\mathbf{j}$, the vector of magnitude 10 at $30°$ above the positive x-axis. [2 marks]

Question 3

Question 3 – Parallel vectors *(Section A · Skills practice)*

Work with parallel vectors.

a. $\mathbf{p} = 3\mathbf{i} + \lambda\mathbf{j}$ is parallel to $\mathbf{q} = -6\mathbf{i} + 8\mathbf{j}$. Find λ. [3 marks]
b. Show that $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 9 \end{pmatrix}$ are parallel and state the ratio of their magnitudes. [3 marks]
c. $\mathbf{a} = 2\mathbf{i} + k\mathbf{j}$ and $\mathbf{b} = (k+1)\mathbf{i} + 6\mathbf{j}$ are parallel. Find the possible values of k. [4 marks]

Question 4

Question 4 – Position vectors and ratios *(Section A · Skills practice)*

Relative to the origin O, A has position vector $2\mathbf{i} + 5\mathbf{j}$ and B has position vector $8\mathbf{i} - \mathbf{j}$.

a. Find and . [3 marks]

b. Find the position vector of the midpoint M. [2 marks]
c. P lies on AB with $AP : PB = 1 : 2$. Find the position vector of P. [3 marks]

Question 5

Question 5 – A first geometric proof *(Section A · Skills practice)*

$OABC$ is a parallelogram with and .

a. Write and in terms of $\mathbf{a}$ and $\mathbf{c}$. [2 marks]

b. Find the position vectors of the midpoints of OB and AC. [3 marks]
c. What does your answer show about the diagonals of a parallelogram? [1 mark]

Question 6

Question 6 – Velocity and speed *(Section A · Skills practice)*

A boat has velocity $3\mathbf{i} + 4\mathbf{j}$ m/s, where $\mathbf{i}$ is east and $\mathbf{j}$ is north.

a. Find its speed and its bearing to 1 decimal place. [3 marks]
b. Find its position after 10 s if it starts at the origin. [1 mark]
c. A second vessel travels at 12 km/h on a bearing of $120°$. Write its velocity in the form $a\mathbf{i} + b\mathbf{j}$ to 2 decimal places. [3 marks]

Question 7

Question 7 – Orienteering course *(Section B · Real-life practice)*

An orienteering course in a forest starts at $S(0, 0)$, goes to control $C_1(3, 4)$, then $C_2(8, 1)$ and finishes at $F(8, -5)$. Coordinates are in kilometres east and north of the start.

Leg	Start	End
1	$S(0, 0)$	$C_1(3, 4)$
2	$C_1(3, 4)$	$C_2(8, 1)$
3	$C_2(8, 1)$	$F(8, -5)$

a. Write down the vector for each leg. [3 marks]
b. Find the total distance run to 2 decimal places. [3 marks]
c. Find the straight-line distance from S to F. [2 marks]
d. Find the bearing of F from S to the nearest degree. [2 marks]

Question 8

Question 8 – Crossing a river *(Section B · Real-life practice)*

A river is 120 m wide and flows east at 3 m/s. A boat can travel at 4 m/s in still water. Let $\mathbf{i}$ be east and $\mathbf{j}$ be north (across the river).

a. The boat steers due north. Find its velocity over the ground, the time to cross, and how far downstream it lands. [4 marks]
b. Find the bearing of the boat\'s actual path. [2 marks]
c. To land directly opposite its starting point the boat steers at an angle α west of north. Find α. [3 marks]
d. Find the time to cross in this case. [3 marks]

Question 9

Question 9 – Aeroplane in a crosswind *(Section B · Real-life practice)*

An aeroplane has an airspeed of 240 km/h. There is a wind of 50 km/h blowing towards the east. Take $\mathbf{i}$ as east and $\mathbf{j}$ as north.

a. The pilot heads due north. Find the ground velocity, the ground speed and the bearing of the path. [4 marks]
b. To fly due north over the ground the pilot heads at a bearing β. Find β to 1 decimal place. [3 marks]
c. Find the ground speed in this case. [2 marks]
d. How long does a 600 km flight take in this case? Give your answer in hours and minutes. [2 marks]

Question 10

Question 10 – Two drones *(Section B · Real-life practice)*

Two drones fly in a field. After t seconds, drone P is at $\mathbf{r}_P = (2+3t)\mathbf{i} + (1+2t)\mathbf{j}$ and drone Q is at $\mathbf{r}_Q = (14-t)\mathbf{i} + (-3+4t)\mathbf{j}$ (metres).

a. Find the velocity of each drone. [2 marks]
b. Find their positions when $t = 2$ and the distance between them. [3 marks]
c. Show that the drones never collide. [4 marks]
d. At what time are the two drones at the same x-coordinate? [1 mark]

Question 11

Question 11 – A zip wire *(Section B · Real-life practice)*

A zip wire runs from the top of a tower A at $(-4, 6)$ to a platform tower B at $(6, -4)$ on a plan (units: tens of metres). A rest platform P divides AB in the ratio $AP : PB = 3 : 2$.

a. Find . [1 mark]

b. Find the position vector of P. [3 marks]
c. Find the length of the whole wire in metres to the nearest metre. [2 marks]
d. The wire costs £48 per metre. Find the cost. [2 marks]

Question 12

Question 12 – Passes on the pitch *(Section B · Real-life practice)*

On a football pitch plan (metres), a player passes the ball from $A(10, 8)$ to $B(22, 17)$ and then to $C(17, 29)$. The ball travels at 15 m/s.

a. Write down and . [2 marks]

b. Find the length of each pass. [3 marks]
c. Find the total time for the two passes. [2 marks]
d. How much longer is the route $A \to B \to C$ than the direct route $A \to C$? [3 marks]

Question 13

Question 13 – Straight cable car *(Section B · Real-life practice)*

Three pylons of a cable car are at $A(1, 2)$, $B(4, 8)$ and $C(10, 20)$ on a map (units: 100 m).

a. Find and . [2 marks]

b. Show that A, B and C are collinear. [2 marks]
c. Find the ratio $AB : BC$. [1 mark]
d. The cable AC costs £18 per metre. Find the length and cost of the cable. [3 marks]

Question 14

Question 14 – Two ships at sea *(Section B · Real-life practice)*

Two ships leave port O at the same time. Ship A travels at 24 km/h on a bearing of 060°. Ship B travels at 18 km/h on a bearing of 150°.

a. Write the position of each ship after 2 hours as a vector in $\mathbf{i}, \mathbf{j}$ form (east, north) to 1 decimal place. [4 marks]
b. Find the distance between the ships after 2 hours. [3 marks]
c. Find the bearing of B from A at this time. [3 marks]

Question 15

Question 15 – A parallelogram proof *(Section C · Problem solving)*

$OABC$ is a parallelogram with and . M is the point on AB with $AM : MB = 1 : 2$ and N is the point on CB with $CN : NB = 1 : 2$.

a. Find and in terms of $\mathbf{a}$ and $\mathbf{c}$. [4 marks]

b. Show that . [3 marks]

c. What does this tell you about the lines MN and AC? [2 marks]

d. Triangles BMN and BAC are similar. State the scale factor and how the answer to part (b) confirms it. [2 marks]

Question 16

Question 16 – Collinear points *(Section C · Problem solving)*

, and , where $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

a. Find and . [3 marks]

b. Show that P, Q and R lie on a straight line. [3 marks]

c. Find the ratio $PQ : QR$. [2 marks]

d. The point S has and lies on the same line. Find k. [3 marks]

Question 17

Question 17 – A boat's track past a lighthouse *(Section C · Problem solving)*

A boat has position vector $\mathbf{r} = (2 + 3t)\mathbf{i} + (-1 + 2t)\mathbf{j}$ km at time t hours. A lighthouse L stands at $(10, 0)$.

t (h)	0	1	2	3
Position	$(2, -1)$	$(5, 1)$	$(8, 3)$	$(11, 5)$

a. Find the speed of the boat. [2 marks]

b. Show that has squared length $13t^2 - 52t + 65$ where B is the boat\'s position. [4 marks]

c. Find the time of closest approach to the lighthouse and the shortest distance. [4 marks]

d. At what time does the boat cross the x-axis (the coastline)? [2 marks]

Question 18

Question 18 – A hiker's closed route *(Section C · Problem solving)*

A hiker walks three legs. Leg 1 is 5 km on a bearing of 030°, leg 2 is 8 km on a bearing of 120° and leg 3 is 6 km on a bearing of 250°.

Leg	East component	North component
1	$5\sin 30^\circ = 2.50$	$5\cos 30^\circ = 4.33$
2	$8\sin 120^\circ = 6.93$	$8\cos 120^\circ = -4.00$
3	$6\sin 250^\circ = -5.64$	$6\cos 250^\circ = -2.05$

a. Check the entries for leg 3 and add the missing totals. [3 marks]
b. Find the total displacement vector. [2 marks]
c. How far is the hiker from the start? Give your answer to 2 decimal places. [2 marks]
d. On what bearing must the hiker walk to return directly? [3 marks]

Question 19

Question 19 – Midpoints of a quadrilateral *(Section C · Problem solving)*

$ABCD$ is any quadrilateral with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. P, Q, R and S are the midpoints of AB, BC, CD and DA.

a. Write the position vectors of P, Q, R and S. [2 marks]
b. Show that and . [4 marks]
c. Prove that $PQRS$ is a parallelogram. [2 marks]
d. If $ABCD$ is a rectangle, what special parallelogram is $PQRS$? Justify. [3 marks]

Question 20

Question 20 – Aircraft separation *(Section C · Problem solving)*

Aircraft P is at the origin at time $t = 0$ and flies with velocity $200\mathbf{i} + 150\mathbf{j}$ km/h. Aircraft Q is at $300\mathbf{i}$ km at $t = 0$ and flies with velocity $-100\mathbf{i} + 140\mathbf{j}$ km/h.

a. Write $\mathbf{r}_P$ and $\mathbf{r}_Q$ at time t hours. [2 marks]
b. Show that . [2 marks]
c. The separation rule requires at least 15 km. Show that the squared distance is $90100t^2 - 180000t + 90000$ and find its minimum to the nearest km. [5 marks]
d. State whether the rule is broken. [1 mark]

Question 21

Question 21 – Challenge: a coastal rescue *(Section D · Challenge)*

A lifeboat leaves harbour H at the origin at 12:00 and travels with constant velocity $\mathbf{v}$ km/h to intercept a yacht. At 12:00 the yacht is at $(12, 16)$ km and moves with constant velocity $(-3\mathbf{i} - 2\mathbf{j})$ km/h. The lifeboat\'s speed is 10 km/h.

a. Write the position of the yacht at time t. [1 mark]
b. The lifeboat reaches the yacht at time T. Show that T satisfies an equation of the form $aT^2 + bT + c = 0$ and find a, b and c. [5 marks]
c. Solve it to find the interception time to the nearest minute and the position of the yacht. [4 marks]